L.I. SEDOV

ANALYTICAL MECHANICS
STABILITY OF MOTION
CELESTIAL BALLISTICS

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PREFACE

The Second All-Union Conference on Theoretical and Applied Mechanics was held in Moscow from 29 January to 5 February 1964, under the joint auspices of the National Committee of the USSR for Theoretical and Applied Mechanics, the Institute of Mechanics of the Academy of Sciences of the USSR, and the Moscow State University im. M.V. Lomonosov.

The work of the conference was directed by an organizing committee consisting of: I.I. Artobolevskii, N.Kh. Arutyunyan, G.I. Barenblatt, L.A. Galin, A.L. Gol'denveizer, G.Yu. Dzhanelidze, A.A. Dorodnitsyn, A.Yu. Ishlinskii, S.V. Kalinin, L.M. Kachanov, M.V. Keldysh, P.Ya. Kochina, L.G. Loitsyanskii, A.I. Lur'e—vice-chairman, Yu.A. Mitropol'skii, G.K. Mikhailov—scientific secretary, N.N. Moiseev, N.I. Muskhelishvili—chairman, A.A. Nikol'skii—vice-chairman, D.E. Okhotsimskii, G.I. Petrov, I.M. Rabinovich, Yu.N. Rabotnov—vice-chairman, V.V. Rumyantsev, L.I. Sedov—first vice-chairman, V.V. Sokolovskii, G.G. Chernyi, D.I. Sherman. Technical secretary of the organizing committee—A.I. Derzhavina.

The work of the conference proceeded in three sections.

Section I. General and applied mechanics (chairman—A.Yu. Ishlinskii, vice-chairmen—A.I. Lur'e, D.E. Okhotsimskii, scientific secretary—V.V. Rumyantsev). Section I comprised the subsections:

I-1. Analytical mechanics and the theory of stability of motion;
I-2. Celestial mechanics;
I-3. Oscillations and control;
I-4. Gyroscopy;
I-5. Theory of mechanisms and machines;
I-6. Problems in teaching of mechanics;

Section II. Fluid mechanics (chairman—L.I. Sedov, vice-chairmen—L.G. Loitsyanskii and G.G. Chernyi, scientific secretary—N.N. Moiseev). Section II comprised the subsections:

II-1. General hydromechanics;
II-2. Aerodynamics and gas dynamics;
II-3. Theory of plasma and rarefied gases;
II-4. Motion of viscous fluids, boundary layers, turbulence and heat transfer;
II-5. Hydrodynamics of multi-component systems;
II-6. Applied fluid dynamics.

Section III. Mechanics of solids (chairman — N.I. Muskhelishvili, vice-chairmen—A.L. Gol'denveizer and Yu. N. Rabotnov, scientific secretary—G.I. Barenblatt). Section III comprised the subsections:

III-1. Theory of elasticity;
III-2. Theory of plasticity;
III-3. Theory of plates and shells;

III-4. Rheology and theory of creep;
III-5. Constructional mechanics;
III-6. Soil mechanics;

Over 2000 members and 3000 guests from all the principal scientific and industrial centers of the country participated in the conference. A total of 560 papers were read in 125 sections and subsection meetings.

Of a total of over 100 survey papers specially commissioned by the organizing committee, those submitted to the organizing committee were included in the prepared "Proceedings" of the conference.

The proceedings of the conference are being published in three parts:*

1. Analytical mechanics. Stability of motion. Celestial ballistics.
2. Oscillations. Gyroscopy. Theory of mechanisms. Fluid mechanics.
3. Mechanics of solids.

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* [The present volume is a translation of Part 1 of the Proceedings.]

ANALYTICAL MECHANICS. STABILITY OF MOTION

I. S. Arzhanykh

DEVELOPMENT OF THE THEORY OF CANONICAL EQUATIONS OF ANALYTICAL MECHANICS

Hamiltonian canonical systems have some remarkable properties of relevance for integration problems. In this connection attempts have been made to extend certain theorems of dynamics to arbitrary systems of ordinary differential equations

$$\frac{dx_k}{dt} = X_k(t, x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n. \qquad (1)$$

These investigations fall into three categories: 1) a method is indicated for reducing system (1) to a Hamiltonian system (this is the approach of Liouville, Lie and Koenigs), 2) integration theorems for Hamiltonian systems are extended to system (1), without reducing it to any specific "canonical" form (this method is applied to Poisson's Theorem in § 2); 3) some new universal canonical structure is proposed for system (1), generalizing the Hamiltonian canonical structure, and having properties which also generalize those of Hamiltonian systems (this approach is described in §§ 3—5).

§ 1. The methods of Liouville and Lie-Koenigs

Along with the variables x_k we introduce the variables y_k and form the function

$$H = \sum_{k=1}^{n} y_k X_k(t, x_1, \ldots, x_n).$$

Equations (1) can then be written in the form

$$\dot{x}_k = \partial H / \partial y_k;$$

if the new variables are determined by the equations

$$\dot{y}_k = -\partial H / \partial x_k, \qquad (2)$$

we obtain a Hamiltonian system. This is Liouville's line of reasoning. We note that system (2) is the conjugate with respect to the variational equations, i.e., the equations define the coefficients of the linear integral invariant

$$\int \sum_{k=1}^{n} y_k \delta x_k. \qquad (3)$$

Lie and Koenigs /1/ start from this point. Let system (1) have the integral invariant (3). We reduce it to canonical form

$$\int \sum_{k_1=1}^{n_1} p_{k_1} \delta q_{k_1}, \quad 2n_1 \leqslant n \tag{3'}$$

and regard the variables p_{k_1} and q_{k_1} as unknowns. Then, introducing other new variables $u_1, u_2, \dots, u_{n-2n_1}$, we obtain a Hamiltonian system

$$\left.\begin{array}{l} \dot{q}_{k_1} = \partial H_1/\partial p_{k_1}, \quad \dot{p}_{k_1} = -\partial H_1/\partial q_{k_1}, \\ H_1 = H_1(t, q_1, \dots, q_{n_1}, p_1, \dots, p_{n_1}) \end{array}\right\} \tag{4}$$

and a supplementary system

$$\dot{u}_s = U_s(t, q_1, \dots, q_{n_1}, p_1, \dots, p_{n_1}, u_1, \dots, u_{n-2n_1}). \tag{1'}$$

Integrating (4) and substituting q_{k_1} and p_{k_1} in (1'), we obtain the system

$$\begin{gathered} \dot{u}_s = V_s(t, u_1, \dots, u_{n-2n_1}), \\ s = 1, 2, \dots, n - 2n_1, \end{gathered} \tag{5}$$

to which we apply the same transformations. We obtain a chain of Hamiltonian canonical systems.

The disadvantage of Liouville's method is that it introduces many superfluous (redundant) unknowns. The Lie-Koenigs method avoids this, but requires knowledge of the integral invariant (3), i.e., we must first find the solutions of the Liouville system (2) as functions of t and x_k. These methods are therefore only of general theoretical interest.

§ 2. Poisson's theorem

The relevance of Poisson's theorem for Hamiltonian canonical equations is quite obvious from the following considerations. If $\varphi = a = \text{const}$ is an integral, then the variational equations have the solutions

$$\delta q_\nu = \varepsilon \partial\varphi/\partial p_\nu, \quad \delta p_\nu = -\varepsilon \partial\varphi/\partial q_\nu, \quad \varepsilon = \text{const}. \tag{6}$$

Consequently, if we have a second integral $\psi = b$, then the integral of the expanded system will be $\delta\psi = \varepsilon c$. Therefore $(\varphi, \psi) = c$ is also an integral of the Hamiltonian system.

For an arbitrary system (1) Poisson's theorem is replaced by Boole's theorem: if there exists an integral

$$\sum_{k=1}^{n} v_k(t, x_1, \dots, x_n)\, y_k = a$$

for the Liouville system and $F = \text{const}$ is an integral of system (1), then another integral of this system is

$$\sum_{k=1}^{n} v_k \frac{\partial F}{\partial x_k} = b. \tag{7}$$

The Liouville system, as we know, is related to the linear integral invariants. It is therefore natural to try to generalize Boole's theorem (and thereby also Poisson's theorem) on the basis of the theory of integral invariants. The first investigations of this type are due to Poincaré /2/. We shall describe a more direct approach.

Suppose system (1) admits m linear integral invariants

$$I_\mu = \int \sum_\nu u_\nu^{(\mu)} \delta x_\nu, \quad \mu = 1, 2, \ldots, m. \tag{8}$$

We denote by $[x_{\nu_1} x_{\nu_2} \ldots x_{\nu_m}]$ the determinant

$$\begin{vmatrix} u_{\nu_1}^{(1)} & u_{\nu_2}^{(1)} & \ldots u_{\nu_m}^{(1)} \\ \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ u_{\nu_1}^{(m)} & u_{\nu_2}^{(m)} & \ldots u_{\nu_m}^{(m)} \end{vmatrix}$$

and set up the skew-symmetric form

$$\{I_1 I_2 \ldots I_m\} \equiv \sum\sum_{\nu_1<\nu_2<\ldots<\nu_m} \ldots \sum \pi_{\nu_1\nu_2\ldots\nu_m} [x_{\nu_1} x_{\nu_2} \ldots x_{\nu_m}], \tag{9}$$

where $\pi_{\nu_1\nu_2\ldots\nu_m}$ is a skew-symmetric tensor satisfying the following system of equations:

$$\frac{d}{dt} \pi_{\nu_1\nu_2\ldots\nu_m} = \sum_{\mu=1}^{m} \sum_{k=1}^{n} \frac{\partial X_{\nu_\mu}}{\partial x_k} \pi_{\nu_1\ldots\nu_{\mu-1} k \nu_{\mu+1}\ldots\nu_m}. \tag{10}$$

Then, as is easily seen, another integral is

$$\{I_1 I_2 \ldots I_m\} = \text{const}. \tag{11}$$

If system (1) has the integrals $F_\mu = c_\mu$, then it is well-known that

$$u_\nu^{(\mu)} = \partial F_\mu / \partial x_\nu \tag{12}$$

and, consequently, we have the integral

$$\sum\sum_{\nu_1<\nu_2<\ldots<\nu_m} \ldots \sum \pi_{\nu_1\nu_2\ldots\nu_m} \frac{\partial (F_1 F_2 \ldots P_m)}{\partial (x_{\nu_1} x_{\nu_2} \ldots x_{\nu_m})} = \text{const} \tag{13}$$

In particular, for the Hamiltonian system

$$\dot{x}_\nu = \partial H / \partial x_{n+\nu}, \quad x_{n+\nu} = -\partial H / \partial x_\nu$$

the components of the tensor π are $\pi_{\nu_1 \ldots \nu_k, n+\nu_1 \ldots n+\nu_k} = 1$, and all the others are zero. Therefore, if this system has the integrals $F_\mu = c_\mu$ ($\mu = 1, 2, \ldots, 2k$)

then the following will also be an integral:

$$\sum\sum\cdots\sum_{\nu_1<\nu_2<\ldots<\nu_k} \frac{\partial(F_1F_2F_3\ldots F_{2k})}{\partial(x_{\nu_1}\ldots x_{\nu_k}x_{n+\nu_1}\ldots x_{n+\nu_k})} = \text{const.} \tag{14}$$

This is Laurent's generalization of Poisson's theorem.

The formulation of the Poisson-Laurent theorem is thus based on the fact that a particular solution of system (10) can be found in explicit form. In some cases the structure of the components of the tensor π for the general system (1) can be indicated. For example, if we have the integrals $G_\lambda = c_\lambda$ $(\lambda = 1, 2, \ldots, n-m)$, then

$$\pi_{\nu_1\nu_2\ldots\nu_m} = \frac{\varepsilon}{M}\frac{\partial(G_1G_2G_3\ldots G_{n-m})}{\partial(x_1\ldots x_{\nu_1-1}x_{\nu_1+1}\ldots x_n)}, \tag{15}$$

where M is the last Jacobi multiplier, and

$$\varepsilon = (-1)^{\sum\limits_{\mu=1}^{m}\nu_\mu}.$$

As an example we mention the Poisson equation system describing the rotation of a heavy rigid body about a fixed point (three classical integrals, $M = 1$).

§ 3. Canonical equations of rank higher than zero

Let us assume that the system in question is of even order $2n$, and write it in the form

$$\left.\begin{aligned} \dot{q}_\nu &= Q_\nu(t, q_1, \ldots, q_n, p_1, \ldots, p_n) \\ \dot{p}_\nu &= P_\nu(t, q_1, \ldots, q_n, p_1, \ldots, p_n) \end{aligned}\right\}. \tag{16}$$

We set up the linear form

$$\omega(\delta) = \sum_{\nu=1}^{n}(Q_\nu\delta p_\nu - P_\nu\delta q_\nu) \tag{17}$$

and reduce it to canonical form

$$\omega(\delta) = \delta H + \sum_{\rho=1}^{r} F_\rho\delta H_\rho. \tag{18}$$

Consequently, our system is represented in canonical form of rank r /3/:

$$\left.\begin{aligned} \dot{q}_\nu &= \frac{\partial H}{\partial p_\nu} + \sum_{\rho=1}^{r} F_\rho\frac{\partial H_\rho}{\partial p_\nu} \\ \dot{p}_\nu &= -\frac{\partial H}{\partial q_\nu} - \sum_{\rho=1}^{r} F_\rho\frac{\partial H_\rho}{\partial q_\nu} \end{aligned}\right\}. \tag{19}$$

The canonical potentials H, H_ρ and the factors F_ρ may be modified by transformations of the form

$$F_l\delta H_l + F_k\delta H_k + \ldots = \delta(F_lH_l + F_kH_k + \ldots) - H_l\delta F_l - H_k\delta F_k - \ldots.$$

An invariant of any such transformation is the quantity

$$I = \sum_\rho\sum_\nu \frac{\partial(F_\rho H_\rho)}{\partial(q_\nu p_\nu)} = -\frac{1}{M}\frac{dM}{dt}. \tag{20}$$

As an example of the reduction of equations of motion to canonical form of rank higher than zero, we consider a holonomic system subjected to nonconservative forces $Q_\nu(t, q_1, \ldots, q_n)$. We represent the elementary work in canonical form

$$\sum_\nu Q_\nu\delta q_\nu = \delta U + \sum_{\rho=1}^{r} V_\rho\delta U_\rho. \tag{21}$$

We then obtain a canonical system of rank r, where

$$H = -T - U + \sum_\nu p_\nu\dot{q}_\nu, \quad F_\rho = V_\rho, \quad H_\rho = U_\rho.$$

Mechanical systems moving under the action of forces $R_\nu(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$, for example, nonholonomic, are easily reduced to canonical form of rank n, where

$$H = -L + \sum_\nu p_\nu\dot{q}_\nu + \sum_\nu q_\nu R_\nu, \quad F_\rho = q_\rho, \quad H_\rho = -R_\rho. \tag{22}$$

The equations of the theory of perturbations in a nonconservative force field also have canonical form of order n (see /4/, p. 564).

For systems (19) Poisson's theorem has a distinctive formulation: if $\varphi = a$ and $\psi = b$ are two integrals of the system, then

$$(\varphi, \psi) + \int_0^t \sum_\rho \begin{vmatrix} (H_\rho, \varphi), & (H_\rho, \psi) \\ (F_\rho, \varphi), & (F_\rho, \psi) \end{vmatrix} dt = c. \tag{23}$$

From the standpoint of the theory of the momentum field /5/, i.e., assuming that

$$p_\nu = p_\nu(t, q_1, \ldots, q_n), \tag{24}$$

the second group of equations of (19) may be represented in the form

$$\frac{\partial p_\nu}{\partial t} + \frac{\partial(H)}{\partial q_\nu} + \sum_\rho F_\rho\frac{\partial(H)}{\partial q_\nu} + \sum_\mu\left(\frac{\partial p_\nu}{\partial q_\mu} - \frac{\partial p_\mu}{\partial q_\nu}\right)\dot{q}_\mu = 0. \tag{25}$$

This is the rotational structure of canonical equations of rank higher than zero. The rotational form of the equations is convenient when generalizing the Hamilton-Jacobi method.

Consider, for example, involution systems, i.e., systems (19) whose principal and supplementary canonical potentials do not depend explicitly on time and are in involution $(H \quad H_\rho) = 0$. $(H_{\rho_1}, H_{\rho_2}) = 0$, $(\rho, \rho_1, \rho_2 = 1, 2, \ldots, r)$. Such systems possess integrals

$$H = h, \quad H_\rho = h_\rho, \tag{26}$$

and for their integration, as follows from equations (25), it is sufficient to determine the total integral $w(q_1, q_2, \ldots, q_n, h, h_1, \ldots, h_r, c_1, \ldots, c_{n-r-1})$ from the combined system

$$\left.\begin{aligned} H(q_1, \ldots, q_n, \ \partial w/\partial q_1, \ldots, \partial w/\partial q_n) = h \\ H_\rho(q_1, \ldots, q_n, \ \partial w/\partial q_1, \ldots, \partial w/\partial q_n) = h_\rho \end{aligned}\right\}. \tag{27}$$

The following will then be integrals of the system (19)

$$p_\nu = \partial w/\partial q_\nu, \quad \partial w/\partial h = t - t_0, \quad \partial w/\partial c_\lambda = d_\lambda \, (\lambda = 1, 2, \ldots, n-r-1), \tag{28}$$

and to obtain the remaining integrals we must integrate the system

$$\frac{d}{dt}\,\frac{\partial w}{\partial h_\rho} = F_\rho \quad (\rho = 1, 2, \ldots, r). \tag{29}$$

§ 4. Invariance under contact transformations

One of the essential properties of canonical systems of rank zero (Hamiltonian systems) is their invariance under contact transformations /1/ (see also /6/, p. 174). Canonical systems of rank higher than zero possess the same property.

We carry out the transformation $(p, q) \to (\eta, \xi)$ according to the formulas

$$\left.\begin{aligned} p_\nu &= \frac{\partial w}{\partial q_\nu} + \sum_{\sigma=1}^{s} \Lambda_\sigma \frac{\partial w_\sigma}{\partial q_\nu} \\ -\eta_\nu &= \frac{\partial w}{\partial \xi_\nu} + \sum_{\sigma=1}^{s} \Lambda_\sigma \frac{\partial w_\sigma}{\partial \xi_\nu} \\ w_\sigma(t, q_1, \ldots, q_n, \ &\xi_1, \ldots, \xi_n) = 0 \end{aligned}\right\}. \tag{30}$$

System (19) is then transformed into a similar system

$$\left.\begin{aligned} \dot{\xi}_\nu &= \frac{\partial Z}{\partial \eta_\nu} + \sum_\rho \Phi_\rho \frac{\partial Z_\rho}{\partial \eta_\nu} \\ \dot{\eta}_\nu &= -\frac{\partial Z}{\partial \xi_\nu} - \sum_\rho \Phi_\rho \frac{\partial Z_\rho}{\partial \xi_\nu} \end{aligned}\right\}, \tag{31}$$

where

$$Z = H + \frac{\partial w}{\partial t} + \sum_\sigma \Lambda_\sigma \frac{\partial w_\sigma}{\partial t}, \quad \Phi_\rho = F_\rho, \quad Z_\rho = H_\rho. \tag{32}$$

The proof of this theorem is the same as that of the analogous theorem for Hamiltonian systems /7/.

The invariance property may be used for extending to system (19) a method similar to the Hamilton-Jacobi method. First we consider an involution system. We carry out the transformation

$$p_\nu = \partial w/\partial q_\nu, \quad -\eta_\nu = \partial w/\partial \xi_\nu \tag{33}$$

and determine the function w from the equations

$$H = \xi_n, \quad H_\rho = \xi_{n-\rho} \quad (\rho = 1, 2, \ldots, r). \tag{34}$$

But then we obtain

$$\left.\begin{array}{l} \dfrac{\partial Z}{\partial \eta_\nu} = 0, \quad \dfrac{\partial Z_\rho}{\partial \eta_\nu} = 0, \quad \dfrac{\partial Z}{\partial \xi_\mu} = 0, \quad \dfrac{\partial Z_\rho}{\partial \xi_\lambda} = 0 \\ \nu = 1, \ldots, n; \quad \mu = 1, \ldots, n-1; \quad \lambda = 1, \ldots, n-r-1 \end{array}\right\} \tag{35}$$

$$\frac{\partial Z}{\partial \xi_n} = 1, \quad \frac{\partial Z_\rho}{\partial \xi_{n-\rho_1}} = \begin{cases} 0, & \text{if} \quad \rho_1 \neq \rho, \\ 1, & \text{if} \quad \rho_1 = \rho. \end{cases} \tag{36}$$

Therefore $\dot{\xi}_\nu = 0$, $\dot{\eta}_\lambda = 0$, $\dot{\eta}_n = -1$, $\dot{\eta}_{n-\rho} = -F_\rho$, or

$$\xi_\nu = \text{const}, \quad \eta_\lambda = \text{const}, \frac{\partial w}{\partial \xi_n} = t - t_0, \quad \frac{d}{dt}\frac{\partial w}{\partial \xi_{n-\rho}} = F_\rho. \tag{37}$$

For the general canonical system (19) we carry out the transformation (30) and chose s so that the system of partial differential equations

$$\left.\begin{array}{l} H + \dfrac{\partial w}{\partial t} + \sum\limits_\sigma \Lambda_\sigma \dfrac{\partial w_\sigma}{\partial t} = 0 \\ F_{\rho_1} = 0, \quad H_{\rho_2} = 0 \quad (\rho_1 \neq \rho_2) \\ \dfrac{\partial w_\sigma}{\partial t} + \sum\limits_\nu \dfrac{\partial w_\sigma}{\partial q_\nu}\left(\dfrac{\partial H}{\partial p_\nu} + \sum\limits_{\rho_1} F_{\rho_1}\dfrac{\partial H_{\rho_1}}{\partial p_\nu}\right) = 0 \end{array}\right\}, \tag{38}$$

where p_ν are replaced by their values from (30), is consistent. We then obtain the integrals $\xi_\nu = \text{const}$, $\eta_\nu = \text{const}$. The number s is called the genus of the momentum field. One of the important problems of the theory of integration of canonical equations of rank higher than zero consists in the preliminary calculation of the genus of the momentum field. For involution systems the genus is zero.

§ 5. Relation of the integration problem to the theory of Lie groups

The following kinetic system of rank r corresponds to the canonical system (19):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot{q}_\nu}\right) = \frac{\partial L}{\partial q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial q_\nu}, \tag{39}$$

where

$$L = -H + \sum_\nu \dot{q}_\nu p_\nu, \quad K_\rho = F_\rho, \quad L_\rho = -H_\rho, \tag{40}$$

and the momenta p_ν must be expressed in terms of $\dot{q}_\nu$ using the first group of equations of (19).

Many mechanical systems are easily represented in the form (39). Consider, for example, the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu} - \frac{\partial T}{\partial q_\nu} = Q_\nu(t, q_1, \ldots, q_n). \tag{41}$$

They can be written in the form (39) by setting $r = n$,

$$L = T, \quad K_\rho = Q_\rho, \quad L_\rho = q_\rho. \tag{42}$$

The equations of motion of nonholonomic systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\nu} - \frac{\partial L}{\partial q_\nu} = \sum_{\mu=1}^{m} M_\mu A_{\nu\mu} \tag{43}$$

also have the form (39), where $K_\rho = \sum\limits_\mu M_\mu A_{\rho\mu}, L_\rho = q_\rho, (\rho = \overline{1, n})$. One often encounters chain systems

$$\left.\begin{array}{l} \frac{d}{dt}\frac{\partial L^{(1)}}{\partial \dot{q}_{\nu_1}} - \frac{\partial L^{(1)}}{\partial q_{\nu_1}} = 0, \quad \nu_1 = 1, 2, \ldots, n_1 \\ \frac{d}{dt}\frac{\partial L^{(2)}}{\partial \dot{q}_{\nu_2}} - \frac{\partial L^{(2)}}{\partial q_{\nu_2}} = 0, \quad \nu_2 = n_1 + 1, \ldots, n_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d}{dt}\frac{\partial L^{(k)}}{\partial \dot{q}_{\nu_k}} - \frac{\partial L^{(k)}}{\partial q_{\nu_k}} = 0, \quad \nu_k = n_{k-1} + 1, \ldots, n, \end{array}\right\} \tag{44}$$

where $L^{(l)} = L^{(l)}(t, q_1, \ldots, q_{n_1}, q_{n_1+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_{n_1}, \dot{q}_{n_1+1}, \ldots, \dot{q}_n)$. They may also be reduced to the form of kinetic equations of rank higher than zero, by representing the linear form

$$\sum_{l=1}^{k}\sum_{\nu_l}\left(\frac{\partial L^{(l)}}{\partial \dot{q}_{\nu_l}}\delta\dot{q}_{\nu_l} + \frac{\partial L^{(l)}}{\partial q_{\nu_l}}\delta q_{\nu_l}\right) \tag{45}$$

in canonical form

$$\delta L + \sum_\rho K_\rho \delta L_\rho.$$

The existence of integrals of (39) which are linear in p_ν, is closely related to the theory of Lie groups.

Suppose the infinitesimal transformations of the group are

$$\vartheta q_\nu = \varepsilon u_\nu(t, q_1, \ldots, q_n) \quad (\nu = 1, 2, \ldots, n) \tag{46}$$

and suppose the following equality is satisfied:

$$\vartheta L + \sum_\rho K_\rho \vartheta L_\rho = 0. \tag{47}$$

Then we have the integral

$$\sum_\nu u_\nu\left(\frac{\partial L}{\partial \dot q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right) = \text{const.} \tag{48}$$

In fact, let us write equality (47) in explicit form

$$\sum_\nu\left[\left(\frac{\partial L}{\partial \dot q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right)\vartheta \dot q_\nu + \left(\frac{\partial L}{\partial q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial q_\nu}\right)\vartheta q_\nu\right] = 0,$$

or

$$\frac{d}{dt}\sum_\nu\left(\frac{\partial L}{\partial \dot q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right)\vartheta q_\nu +$$

$$+\sum_\nu\left[\frac{\partial L}{\partial q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial q_\nu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\nu} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right)\right]\vartheta q_\nu = 0. \tag{49}$$

But by virtue of the equations of motion (39) the second sum vanishes.

Equality (47) may be satisfied due to the invariance of the principal and supplementary kinetic potentials ($\vartheta L = 0, \vartheta L_\rho = 0$), or due to the equalities

$$\vartheta L = 0, \quad \sum_\rho K_\rho \vartheta L_\rho = 0. \tag{50}$$

If we apply this case to the system (41), we see that if the kinetic energy is invariant with respect to the Lie group, and the work of the generalized forces on the displacements ϑq_ν is zero, then the system admits an integral which is linear in the momenta.

Applied to the nonholonomic system (43), we have the following sufficient conditions for the existence of a linear integral:

$$\vartheta L = 0, \quad \sum_\nu A_{\nu\mu}\vartheta q_\nu = 0, \quad (\mu = 1, 2, \ldots, m). \tag{51}$$

A mixed equation system

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_\mu} + \sum_\rho S_\rho \frac{\partial R_\rho}{\partial \dot q_\mu}\right) = \frac{\partial R}{\partial q_\mu} + \sum_\rho S_\rho \frac{\partial R_\rho}{\partial q_\mu} \quad (\mu = 1, 2, \ldots, m),\\
&\dot q_{m+\lambda} = \frac{\partial R}{\partial p_{m+\lambda}} + \sum_\rho S_\rho \frac{\partial R_\rho}{\partial p_{m+\lambda}},\\
&\dot p_{m+\lambda} = -\frac{\partial R}{\partial q_{m+\lambda}} - \sum_\rho S_\rho \frac{\partial R_\rho}{\partial q_{m+\lambda}} \quad (\lambda = 1, 2, \ldots, n-m),
\end{aligned}\right\} \tag{52}$$

where

$$R(t, q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_m, p_{m+1}, \ldots, p_n) =$$

$$= -L + \sum_\lambda p_{m+\lambda}\dot q_{m+\lambda}, \quad R_\rho = -L_\rho, \quad S_\rho = K_\rho,$$

$$p_{m+\lambda} = \frac{\partial L}{\partial \dot q_{m+\lambda}} + \sum_\rho K_\rho \frac{\partial L_\rho}{\partial \dot q_{m+\lambda}} \tag{53}$$

also corresponds to a canonical system of rank higher than zero. Let the functions R and R_ρ be invariant with respect to a group with an infinitesimal operator

$$\sum_{\mu=1}^{m} v_\mu (t, q_1 q_2, \ldots, q_n;\ p_{m+1}, \ldots, p_m) \frac{\partial}{\partial q_\mu} . \qquad (54)$$

Then the system (52) admits the integral

$$\sum_{\mu} v_\mu \Big(\frac{\partial R}{\partial \dot{q}_\mu} + \sum_{\rho} S_\rho \frac{\partial R_\rho}{\partial \dot{q}_\mu} \Big) = \text{const.} \qquad (55)$$

If the coordinates $q_{m+\lambda}$ are cyclic, we have the integrals $p_{m+\lambda} = \text{const}$.

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V.I. Arnol'd

STABILITY AND INSTABILITY OF DYNAMIC SYSTEMS WITH MANY DEGREES OF FREEDOM

§ 1. **Introduction.** Nearly integrable conservative systems are often encountered in mechanics (the many-body problem, the rapidly spinning top, etc.). The study of such systems often utilizes the approximate, asymptotic methods of perturbation theory. In the present paper we consider the correspondence between exact and approximate solutions of the equations of dynamics over an infinite time interval.

It turns out that the stability of systems with one or two degrees of freedom may usually be ascertained on the basis of the first approximation. On the other hand, a system with $s > 3$ degrees of freedom may be stable in all the approximations of perturbation theory, and unstable in reality.

We start with an example of an oscillatory system with one degree of freedom.

§ 2. **Example.** Consider the motion of a pendulum, described by the equation

$$\ddot{x} + \omega^2 \sin x = \varepsilon f(x, t),$$

where the disturbing force f has period 2π in the angle x and the time t. The motion of the pendulum is described by a "trajectory" $x(t), \dot{x}(t)$ in the three-dimensional "phase space" $x, \dot{x}, t$. Consider a point $X = (x, \dot{x})$ in the plane of the initial conditions $t = 0$. The trajectory issuing from this point intersects the plane $t = 2\pi$ at the point $TX = (x(2\pi), \dot{x}(2\pi))$. By virtue of the periodicity in t, the planes $t = 0$ and $t = 2\pi$ can be identified. Thus, the trajectory is described on the "secant plane" $x, \dot{x}$ by the sequence of points $X, TX, T^2X, \ldots$, where T is the transformation mapping X onto TX.

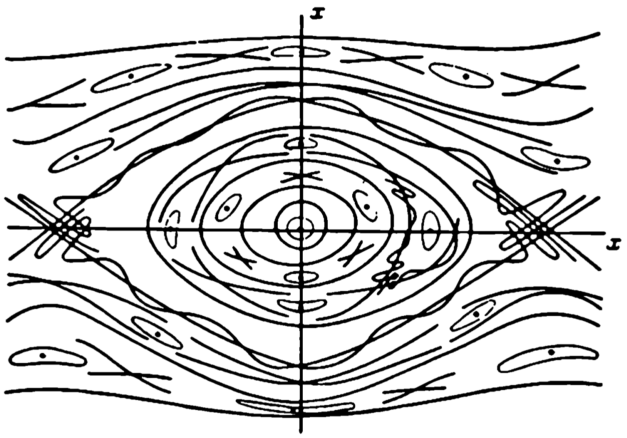
First let $\varepsilon = 0$. Then the energy $E = {}^1/_2\dot{x}^2 + \omega^2(1 - \cos x)$ is conserved, and all the points T^nX lie on one curve $E = \text{const}$. For small energies $(E < 2\omega^2)$, this curve is closed. One may define the "frequency" $\Omega(E)$ as the limit $\lim\limits_{n\to\infty} N(n)/n$, where $N(n)$ is the number of revolutions of the point T^nX around the point $x = \dot{x} = 0$. It is obvious that $\Omega(E) = \omega$ only if $E = 0$ (nonlinearity).

We call the curve $E = \text{const}$ a resonance curve if $\Omega(E) = m/n$ where m and n are integers. If the point X lies on a resonance energy level, then $T^nX = X$, and the sequence $X, TX, \ldots$ consists of a finite number of different points. At nonresonance energy levels, $T^nX \neq X$ and the sequence $X, TX, \ldots$ is dense in the curve $E = \text{const}$.

Now let $\varepsilon \neq 0$. In this case the energy E is not conserved and the points T^nX do not lie on the curves $E = \text{const}$. It turns out that for small ε, the pattern described above is deformed as follows. Most of the nonresonance curves $E = \text{const}$ give rise to almost closed curves Γ which are invariant under the perturbed transformation T. For a point X lying on such a curve Γ, the sequence T^nX is dense in Γ.

In the presence of perturbation the resonance curves disintegrate. Instead of the curve $\Omega(E) = m/n$, there are $2kn$ points X for which $T^nX = X$. Of these points half are stable (elliptic) and half unstable (hyperbolic). The resulting pattern is depicted schematically in Figure 1. Note the complicated behavior of the separatrices of the hyperbolic points.

The existence of the invariant curves Γ implies, in particular, the stability of the pendulum's oscillations: for a sufficiently small perturbation ε, the change $E(t) - E(0)$ in energy remains small for all t, $-\infty < t < +\infty$.



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The curves Γ are found using a version of perturbation theory suggested by Kolmogorov in 1954. This method is similar to Newton's method of tangents for the solution of algebraic equations. The first approximation coincides with the first approximation of asymptotic methods. It is well-known that the system of succesive approximations of perturbation theory diverges (small denominators). In Newton's method the error in each approximation is of the order of the square of the error in the previous one. This rapid convergence overcomes the influence of the small denominators and the method yields converging approximations.

§ 3. Some results. Since 1954 the indicated method has been developed by the author, J. Moser, and others. We shall not give a detailed summary of all the results,* but only a few typical ones.

A. Let T be an area-preserving mapping of the plane onto itself (the example of § 2) given in polar coordinates r, φ by the formulas

$$\begin{pmatrix} r \\ \varphi \end{pmatrix} \to \begin{pmatrix} r + R(r,\varphi) \\ \varphi + \lambda + \lambda_1 r^2 + \Phi(r,\varphi) \end{pmatrix} \text{where} \begin{matrix} |R| < Cr^4 \\ |\Phi| < Cr^3. \end{matrix}$$

If $\lambda \neq 2\pi m/3$, $2\pi m/4$ and $\lambda_1 \neq 0$, then the fixed point 0, 0 is stable in the Lyapunov sense.

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[*] For references see Arnol'd, V.I. Malye znamenateli (Small Denominators).—Uspekhi matematicheskikh nauk, Vol. 18, No. 6 (114): 91—192. 1963.

The theorems of the stability of the equilibrium positions and the periodical solutions of Hamiltonian systems with two degrees of freedom are similar in form. These theorems imply, in particular, the stability of the Lagrangian of the periodic motion of the restricted three-body problem.

B. Consider a heavy rigid body fixed at an arbitrary point 0 and rotated rapidly. It turns out that the magnitude and inclination to the horizontal of the angular-momentum vector always remain near their initial values. In particular, if the body is rotated rapidly about the major or minor inertia axis, then the axis of rotation in the body will always remain near that axis (in space it will precess slowly around the vertical).

C. Another result is the perpetual adiabatic invariance of the action variable in the case of slow periodic variation of the parameters of a nonlinear system. Consider, for example, motion in a potential well:

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y},$$

where

$$V = (1 + \varepsilon^2 x^2)^2 \frac{y^2}{2}.$$

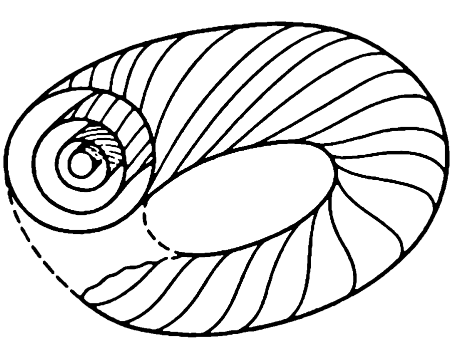
At the "bottom" of the well the particle may escape to infinity, if $y = \dot{y} = 0$. It turns out that for small ε the well traps particles with $y^2 + \dot{y}^2 \neq 0$; $\sup|x|(t) < \infty$. This follows from the perpetual adiabatic invariance of

$$I_y = \frac{\dot{y}^2 + (1 + \varepsilon^2 x^2)^2 y^2}{2(1 + \varepsilon^2 x^2)}.$$

The increment $|I_y(t) - I_y(0)|$ is small for all t, $-\infty < t < +\infty$.

§ 4. Multi-frequency motion. All previous examples refer to systems with two degrees of freedom. The proofs are based on finding multi-frequency conditionally periodic motions. Such motions also exist in systems with many degrees of freedom.

Consider, in particular, the three- (or many-) body problem. If the masses of the "planets" are sufficiently small relative to the mass of the central body, the motion is Keplerian in the first approximation. Let us assume that the initial conditions are such that the eccentricities and inclinations of the Keplerian ellipses are small, and the bodies move along the ellipses in one direction.



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It turns out that for most initial conditions of the indicated type the motion is conditionally periodic (in particular, it is bounded in an infinite time interval).

The use of conditionally-periodic solutions in order to ascertain the stability of all motions is based on simple topological considerations. Consider a conservative system with s degrees of freedom and a $2s$-dimensional phase space. In this space an s-frequency motion is represented by the winding of a Δ-dimensional invariant torus M (one can visualize the case $s = 2$—an ordinary torus with angular coordinates φ_1 and φ_2 and motion $\dot\varphi_1 = \omega_1$, $\dot\varphi_2 = \omega_2$ with frequencies ω_1 and ω_2). The energy integral divides the $2s$-dimensional phase space into $2s - 1$-dimensional invariant manifolds $E = \text{const}$. If $s = 2$ the two-dimensional invariant toruses M lie in a three-dimensional energy level $E = \text{const}$, resembling coaxial tubes (Figure 2). They partition the manifold $E = \text{const}$, though not filling it completely (Figure 1). Trajectories in the gaps between the toruses M cannot intersect them and always remain between them. In systems with two degrees of freedom, stability therefore follows from the existence of the toruses M.

§ 5. Instability. Now let the number of degrees of freedom be $s > 3$, for example, $s = 3$. The energy level is of dimension $2s - 1 = 5$, and the invariant toruses M of dimension $s = 3$. But a three-dimensional manifold does not partition a five-dimensional one (just as a point does not partition a plane or a straight line a space). The gaps between the invariant toruses M therefore communicate with each other. In the region consisting of these gaps instability may occur.

In fact, consider a system with angular coordinates φ_1 and φ_2,

$$\ddot\varphi_1 = -\frac{\partial V}{\partial\varphi_1}, \quad \ddot\varphi_2 = -\frac{\partial V}{\partial\varphi_2}, \quad \text{where } V = \varepsilon(\cos\varphi_1 - 1)[1 + \mu(\sin\varphi_2 + \cos t)].$$

Here the phase space $\varphi_1, \varphi_2, \dot\varphi_1, \dot\varphi_2, t$ is five-dimensional, ε and μ are small parameters, and the unperturbed motion has three-frequencies ($\dot\varphi_k = \omega_k$, $\dot t = 1$).

The quantity $\dot\varphi_2 = I_2$ does not have secular variations in all the approximations of perturbation theory. It turns out that for arbitrarily small ε, μ, α there are initial conditions for which $I_2(0) < 1$, $I_2(t) > 1/\alpha$. These initial conditions are situated in the resonance zone $\varphi_1 = I_1 \sim \sqrt{\varepsilon}$. The secular variation of I_2 is not revealed by asymptotic methods, because its rate is of order $\exp(-1/\sqrt{\varepsilon})$.

G. V. Aronovich and N. A. Kartvelishvili

APPLICATION OF STABILITY THEORY TO STATIC AND DYNAMIC STABILITY PROBLEMS OF POWER SYSTEMS

1. The problem of stability as a constituent part of the problem of reliability and economy of a power system

The problem of power system reliability is currently regarded, on the one hand, as that of determining the necessary power and energy reserves and, on the other, as that of the stability of the steady-state conditions of the system. The solutions of these two problems are applied almost independently of each other, and no appropriate allowance is made for the daily and seasonal variability of the operating conditions of the system. Until recently (and in the overwhelming majority of the literature also currently) determination of the emergency reserve has been based on the probability of accidental loss of generated power by the so-called unit failure probability p. The inadequacy of such an approach is clear from the fact that p, i.e., the ratio of the total idle time of the unit to its total exploitation time, does not indicate the frequency of failures and their daily and seasonal distribution. For instance, the different effects of night and day failure, or, if the system contains a hydroelectric station, of failures in the high-water period and low-water period, is obvious.

The stability problem consists in determining the stability or instability of the system for given values of its parameters under a given perturbation, or in determining values of the parameter which ensure stability, without determining the probabilities of stability violation. The formulas given in /1/ for the probability of stability violations do not solve the problem (even if one agrees with the underlying assumptions), since in general the problem of estimating this probability cannot be effectively formulated and solved by the theory of random events, from which standpoint it is considered in /1/, but needs the more advanced theory of stochastic processes.

From the viewpoint of the reliability of electric power supply to consumers, there is no essential difference between stability violations and other failures, and these violations should be taken into account in estimating the total probability of continuous electric supply on a par with failures in stationary units, breaks in electric transmission lines and so on. The reliability problem is, in the final analysis, the problem of the economy of the power system, since a failure in power supply to consumers entails commercial loss or inconvenience for the public, but, with extremely rare exceptions, does not have disastrous results. In operation this problem is an organic part of the problem of the economically most advantageous operating conditions of the system, and, at the planning stage, of the problem of the most advantageous configuration of the system, parameters of the stations and transmission lines.

The restrictions, natural or otherwise, governing the control functions x_i of the system's operating conditions (such as the loads on the stations, the current in the transmission lines, the voltages at the junctions of the network, the filling of the water reservoirs of the hydroelectric station, etc.) may be formulated as inequalities

$$F(x_1, \ldots, x_m; \quad y_1, \ldots, y_n, \quad a_1, \ldots, a_r) > 0. \tag{1.1}$$

Here $y_1, \ldots, y_n$ are random functions of time expressing the external conditions (natural discharge into the water reservoirs of the hydroelectric station, the loads on the supply transformers etc.), and $a_1, \ldots, a_r$ are the parameters of the system. The simplest of inequalities (1.1) express obvious circumstances, such as the fact that the load on the stations cannot exceed the available power or that the water level in the reservoir lies between the dead storage level and the normal backwater level.

The most complicated inequalities refer to stability. Failure of equipment, short circuits, nonselective operation of protective devices, etc., lead to a situation in which the parameters a_i do not remain constant, but are random time functions.

The combined problem of economy and reliability at the operating stage consists in regulating the choice of control functions x_i to ensure minimum mathematical expectation of the annual operating expenditures subject to satisfaction of inequalities (1.1). That part of these expenditures which depends on the choice of a strategy consists of the cost of fuel for the thermal power plants of the system and of losses due to deficient power supply to consumers.

The economy and reliability problem is considered in this formulation in /2—4/, though not all the inequalities (1.1) are taken into account.

2. Inequalities relating to stability

The literature on stability of power systems uses terms /5/ to which correspond the following concepts of general stability theory.

1. Static stability—local stability (under infinitesimal perturbations of the steady-state conditions).
2. Dynamic stability—global stability (under finite perturbations of conditions to be established upon termination of the perturbation).
3. Resultant stability—asymptotic stability.
4. Synchronous dynamic stability—Lyapunov stability, ordinary (for example, in the conservative idealization case) or asymptotic, under perturbations which do not cause deviations of more than 2π in the angular coordinates (determining the positions of the rotors of the units) from their values in conditions established after termination of the perturbation.
5. Stability with respect to a given station—stability in a given coordinate.

These definitions assume the power system to be an autonomous dynamic system.

Henceforth, dynamic stability will always mean ordinary asymptotic stability, since the difference between synchronous stability and resultant stability is not essential.

Those inequalities of (1.1) relating to stability always express, in particular, the conditions for static stability of a system. In the presence of random perturbations, the conditions for static stability under a load equal to the mathematical expectation of the fluctuating load must be satisfied.

If there are weak links in the system, uncontrollable power currents through these links become appreciable. This is due to the fact that automatic control in power systems responds to relatively slow power fluctuations, but not to rapid oscillations. The amplitude of the latter is commensurate with the capacity of the weak link, and these oscillations should be regarded as a probabilistic process with unbounded distributions. Here we have a case when, under constant perturbations, the ordinary well-developed theory of stability, which assumes them to be bounded, is inapplicable.

Let us assume that there are no constantly operating random perturbations. Until a short circuit occurs the vector a, whose components are the parameters a_i in any of inequalities (1.1), has the value $a^{(0)}$. Due to the short circuit, this vector assumes the value $a^{(1)}$, and after the damaged line is repaired — the value $a^{(2)}$. The system then reaches new steady-state conditions if $a^{(2)} \in L$, where L is a set determined by the dynamic stability and depending on $a^{(0)}$, $a^{(1)}$, and the time required to repair the short circuit. If $a^{(2)} \notin L$, trouble will develop, which may, for example, cause the system to split. Following repair, the left-hand side of (1.1) will be different in the cases $a^{(2)} \in L$ and $a^{(2)} \notin L$. Introducing a random variable B defined by $B = 1$ in the first case and $B = 0$ in the second case, we may write (1.1) as a single inequality for both cases.

The time elapsing from the occurrence of the short circuit to practical stabilization of the system after its disconnection (the probability of a second short circuit, or of any other failure, during this time may be assumed zero) is insignificant from the economic viewpoint and the transition from $a^{(0)}$ and $a^{(2)}$ may be considered instantaneous. Some time after this transition the vector a assumes a new value, due to another short circuit or to liquidation of the aftereffects of the stability violation caused by the first short circuit, and so on. Thus, the reliability problem, as far as stability is concerned, consists, first, in establishing inequalities (1.1) on the basis of static and dynamic stability calculations, and second, in determining the probability characteristics (distribution functions) of the discontinuous stochastic process of changes in the time vector function a. In this case it is necessary, of course, to take into account that the changes are due not only to short circuits but also to other causes, such as disconnection of certain units (for example, lines) for repairs or gradual changes in the consumer load. In the presence of constantly operating random perturbations, the vector a will be influenced by stability violations caused by these perturbations; this must be reflected in the appropriate calculations.

3. Stability problems for simple systems

There are two principal approaches to the study of the stability of power systems: 1) rigorous (exact or approximate) solution of stability problems for simple power systems (e.g., a generator working through a line into receiving D.C. busbars, systems consisting of two or three generators of commensurate power, etc.); 2) intuitive engineering solutions for multimachine systems, which consist in reducing them to simpler ones. The intuitive character of the second approach is due to the great complexity (mainly the multidimensionality) of the corresponding problems.

The first approach is valuable in that it reveals the essential features of the transient processes, which are characteristic in many cases not only for simple systems but also for general systems. Apparently, their value is also preserved when rigorous approaches to multimachine systems can be found.

The first approach is based on the actual possibility, under certain conditions, of decomposing the total transient process in a power system into components of different physical nature. This in turn makes it possible, when these conditions are satisfied, to consider independently the stability of hydraulic processes in the water intake of a hydraulic power plant (in the pressure of so-called ideal regulators), electromechanical processes (when the power of the primary engines is constant), the control stability of the turbines. In this connection, the past decade has seen the publication of a few results obtained by global consideration of the stability of hydraulic and electromechanical processes in simple dynamic power systems. A review of the corresponding papers is given in /6/.* Here we consider the most important results to appear since /6/.

In 1961 a paper appeared /7/ which considers a conservative idealization of the problem of global stability of the parallel operation of three synchronous generators (the research was carried out mainly in 1945, but was not published at the time). The paper is based on considerations of A.A. Gorev concerning the influence of finite perturbations on the stability of a given system /8/ and the solution uses methods of bifurcation theory. The content of the paper is as follows (in view of the extreme brevity of /7/ it is given almost literally).

According to A.A.Gorev, the equations of motion of three parallel-working machines whose configuration is given by the angles $\theta_1, \theta_2, \theta_3$ can be written in the form of the Lagrange-Maxwell equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right)-\frac{\partial U}{\partial \theta_i}=0, \quad (i=1, 2, 3), \tag{3.1}$$

where

$$\left.\begin{aligned} T &= \frac{1}{2}\left(k_1\dot{\theta}_1^2+k_2\dot{\theta}_2^2+k_3\dot{\theta}_3^2\right),\\ U &= p_1\theta_1+p_2\theta_2+p_3\theta_3+q_{12}\cos(\theta_1-\theta_2)+q_{13}\cos(\theta_1-\theta_3)+\\ &\quad+q_{23}\cos(\theta_2-\theta_3). \end{aligned}\right\}. \tag{3.2}$$

Here p_i and q_i are constants depending on the operating conditions of the system, and

$$p_1+p_2+p_3=0. \tag{3.3}$$

(This condition states that the total power fed into the system through the shafts of the machines is equal to the total power consumed in the network.)

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* See also /19, 20/.

Using the first integral of system (1)

$$k_1\dot{\theta}_1 + k_2\dot{\theta}_2 + k_3\dot{\theta}_3 = \text{const} \tag{3.4}$$

and the substitutions

$$\varphi = \theta_1 - \theta_3, \quad \psi = \theta_2 - \theta_3 \tag{3.5}$$

the problem is reduced to a conservative system with two degrees of freedom, for which the kinetic and potential energies are respectively

$$\begin{gathered} T = \frac{2}{k_1 + k_2 + k_3}\,[k_1k_2(\dot{\varphi} - \dot{\psi})^2 + k_3(k_1\dot{\varphi}^2 + k_2\dot{\psi}^2)], \\ -U = p_1\varphi + p_2\psi + q_{12}\cos(\varphi - \psi) + q_{13}\cos\varphi + q_{23}\cos\psi. \end{gathered} \tag{3.6}$$

The coordinates φ^* and ψ^* of the equilibrium state of this new system, which correspond to rotation of the machines with constant mismatch angles, can be found from the conditions

$$\begin{gathered} \frac{\partial U}{\partial \varphi} = 0, \\ \frac{\partial U}{\partial \psi} = 0. \end{gathered} \tag{3.7}$$

The solution $\varphi = \varphi^*, \psi = \psi^*$ of system (3.7) corresponds to a locally stable equilibrium state if the function U has a maximum at the point φ^*, ψ^*.

Suppose the system is in a stable stationary state and then the parameters p_1, p_2, q_{12}, q_{13} and q_{23} assume new values instantaneously (or during some finite time interval). Then the mismatch angles $\varphi = \varphi(t)$ and $\psi = \psi(t)$ either remain bounded (stable transition), or at least one of them will increase without limit (unstable transition).

This problem is solved by a method indicated in /8/, similar to that of Hill in his well-known investigations of lunar motion /9/ (in an idealization corresponding to the restricted three-body problem).

If $h_1 = T - U$ is the energy of the system immediately after the transition, then, since the system is conservative, the subsequent motion of the system also satisfies

$$-U \leqslant h_1. \tag{3.8}$$

Inequality (3.8) defines a certain domain of values for the variables φ and ψ. For a transition to be stable it is sufficient that the connected component of the domain containing the point φ^*, ψ^* be bounded immediately after the transition. The boundary of this region is the curve

$$U(\varphi, \psi) = h_1. \tag{3.9}$$

The problem is to determine the dependence of these curves on the parameters p_i and q_i.

Since φ and ψ are angular variables, the set of values of φ and ψ is a two-dimensional torus. The problem thus reduces to studying the partition of the torus surface by the family of curves

$$H(\varphi, \psi) \equiv \alpha\varphi + \beta\psi + \cos(\varphi - \psi) + \gamma\cos\varphi + \delta\cos\psi = h, \tag{3.10}$$

where $\alpha = p_1/q_{11}$ and so on (see (3.6)).

These curves are the phase trajectories of the system of equations

$$\begin{aligned} \frac{d\varphi}{d\tau} &= \frac{\partial H}{\partial\psi} = \beta - \sin(\varphi - \psi) - \delta\sin\varphi, \\ \frac{d\psi}{d\tau} &= -\frac{\partial H}{\partial\varphi} = -\alpha + \sin(\varphi - \psi) + \gamma\sin\varphi. \end{aligned} \tag{3.11}$$

For a conservative system a qualitative picture of the partition of the torus surface into trajectories can be found using of the energy integral, as in the case of the phase plane. In accordance with (3.10), this integral can be represented as

$$H(\varphi, \psi) = \alpha\varphi + \beta\psi + G(\varphi, \psi) = \text{const}, \tag{3.12}$$

where α and β are constants, and $G(\varphi, \psi)$ is a periodic function of φ and ψ with period 2π.

The partition of the torus into trajectories forms a number of cells of the same type as in the plane, whose boundaries are the separatrices of the saddle points, and one cell of a new type $D_{\alpha,\beta}$. The cell $D_{\alpha,\beta}$ consists of closed trajectories of the type (m, n), if α and β are commensurable and $\alpha/\beta = m/n$ (m and n being integers), and of nonclosed Poisson-stable trajectories, if α and β are not commensurable (periodic motion of type (m, n) means that in moving along the corresponding closed trajectory, the variable φ changes by $2\pi m$, and the variable ψ by $2\pi n$). The cell $D_{\alpha,\beta}$ is a torus with a number of "holes".

To study the behavior of a conservative dynamic system on a torus as dependent on the parameters, we may restrict ourselves to bifurcations which modify the structure of the torus partition into elementary cells. Such bifurcations occur in the following two cases:

1. The system

$$\frac{\partial H}{\partial\varphi} = 0, \quad \frac{\partial H}{\partial\psi} = 0 \tag{3.13}$$

has roots φ^*, ψ^* for which

$$q = \left\{\frac{\partial^2 H}{\partial\varphi^2}\frac{\partial^2 H}{\partial\psi^2} - \left(\frac{\partial^2 H}{\partial\varphi\,\partial\psi}\right)^2\right\}_{\substack{\varphi=\varphi^* \\ \psi=\psi^*}} = 0.$$

2. The system of equations (3.13) has two solutions: φ_1^*, ψ_1^* and φ_2^*, ψ_2^* for which $H(\varphi_1^*, \psi_1^*) = H(\varphi_2^*, \psi_2^*)$.

It is pointed out in /7/ that this makes it possible to find out all possible modes of partition of the torus into trajectories for arbitrary values of α, β, γ, δ in equations (3.11), and to find domains in the parameter space corresponding to each of these modes. In turn, this implies a number of general statements as to which finite perturbations lead, under conditions of conservative idealization, to unlimited increase in the mismatches of the machine operation, and which keep these mismatches finite.

Recently some other papers have appeared (particularly, /10—12/) in which global stability of a conservative system of three parallel-operating generators is determined, as by Gorev, using the first integral of the equations of motion. However, these papers essentially repeat some results of /8/. The paper /12/, which sets forth a graphical method simplifying the corresponding calculations, is of some interest.

The paper /13/ studies global stability of the nonlinear nonconservative problem of energetics. It considers a salient pole generator working (through a step-up transformer and transmission line) into the busbars of a receiving system of infinite power. It is assumed that the excitation of the generator is controlled by a strong-action regulator. The phase space of the problem is three-dimensional. Using the smallness of certain parameters (corresponding, under the conditions of the problem, to a small degree of nonconservativity), the author establishes an approximate relation between the phase variables, after which he passes from the phase space to the phase plane.

The continuation of /13/ deals with the construction of the attraction region of a stable equilibrium state in the plane. Leaving the details aside, we note that, the results of /13/ notwithstanding, the problem of complete qualitative study of the behavior of nonconservative power systems with a three-dimensional phase space is still essentially open (especially if the system is not almost conservative).

The only, but unfortunately incorrect, attempt to formulate and solve the stability problem in the case of continuously acting random perturbations is made in /14, 15/.

The problem may be formulated as follows. Let z_i $(i = 1, \ldots, n)$ be the generalized coordinates of the system under consideration, $z_i = f_i(t)$ the solution of the equations of the system in the absence of perturbations, and $z_i = z_i(t)$ the solution in the presence of random perturbations whose mathematical expectation is assumed zero. If the condition $[z_1(0) - f_1(0)]^2 + \ldots + [z_n(0) - f_n(0)]^2 < \varepsilon^2$ is satisfied, then for $t \leqslant t^*$ the condition $[z_1(t) - f_1(t)]^2 + \ldots + [z_n(t) - f_n(t)]^2 < \varepsilon^2$ is satisfied. Here t^* is a random variable. It is required to determine the distribution function of t^* for given ε and random initial conditions, when the quantities $z_1(0), \ldots, z_n(0)$ are given by their joint distribution. In practical calculations, ε is determined by technical considerations.

4. Reduction of complex to simple systems

The advent of computers has made it possible to solve the equations of transients (numerically, on digital computers, by simulation, on analog computers) for given initial conditions, i.e., to make a direct check of the stability under a given perturbation for fairly complex systems. Nevertheless, the second method indicated at the beginning of section (3) has not been superseded, since concrete systems with hundreds and thousands of generators are replaced not by simple systems but by computation schemes with tens of generating units. This is described, for example, in /16/ for the Swedish power system. In such cases the same computer programs are usually used for both dynamic (global) and static (local) stability calculations, where they obviate tedious operations with characteristic determinants.

Up to the present a fair number of investigations on the "equivalentization" of power system units, i.e., replacing several units by a single "equivalent" machine, have been described. Such a replacement is in order when the machines being replaced and their controllers are identical, equally loaded and occupy the same position in the system, and the perturbations in question do not cause mutual oscillation of the machines. But even if these conditions are satisfied, one must be sure that the system is stable under infinitesimal perturbations of the individual machines being replaced by a single equivalent one. That stability of this kind may be neglected has become apparent in practice, for example, in the adjustment of the strong-action excitation regulators (i.e., operating according to some functional of the regulated quantity), of the Volga Hydroelectric Station im. Lenin.

In all other cases "equivalentization" of units, i.e., simplification of the network of the power system under consideration, is a somewhat arbitrary method. It is more natural, correct, and effective not to attempt to simplify the network of the power system, but to simplify the equation system of the transients; this does not necessarily imply its reduction to a system of equations describing analogous cases in a simpler power system. Such a reduction is meaningful only for simulation, in the precise sense of the word, of power systems by physical models. In practice, however, physical models of power systems usually play the role of stands for testing equipment, which require not exact similarity but only plausible reproduction of natural conditions.

During the past decade papers have been published in which linearized systems of equations of transients in parallel operating machines are simplified, i.e., static stability is considered. These papers, which make use of the specific properties of the linear equations of concrete problems, will not be considered here.

One general method for simplifying the initial equation system of a complex power system, not connected with the specific features of linear systems, consists in separating the motions of the power system into rapid and slow. The mathematical aspect of this method is the theory of differential equations with small coefficients; the literature of the subject is extensive /6, 17/. The method is widely applied in oscillation theory, and there is no need to dwell on its characterization. Its applications to certain power problems are considered in /6/.

Recently this method has yielded a practically important result in the problem of interaction of electrical, mechanical, and hydraulic processes in power systems and electric stations /18/; it has been shown that within a wide range of operating conditions for a hydraulic power plant in a power system, one can achieve convergence to equilibrium under infinitesimal and finite perturbations, of transients in the water intake installations, for any parameters of these installations, by suitable adjustment of the velocity controllers of the turbines. In other words, hydraulic stability does not entail expensive construction work, but only adjustment measures which do not require additional expenditure.

In this problem the slowest motions—hydraulic oscillations—are considered. In investigating the faster electromechanical oscillations, separation of motions according to their velocities is not always effective. This is clear from the example of a symmetrical power system in the form of a regular polygon with identical generators at the vertices, and loads at the centers of the sides. In this case the differences between the corresponding generalized coordinates of neighboring generators may be neglected (if the perturbation does not lead to sharp deviations in these coordinates at the beginning of the transient process); the small distance between the generators may vary no faster than the coordinates themselves.

Suppose the system of initial equations is reduced to normal Cauchy form

$$\frac{dz_i}{dt} = \frac{1}{T_i} \varphi_i(z_1, \ldots, z_n). \qquad (4.1)$$

Hence,

$$\frac{d}{dt}(z_i - z_k) = \frac{1}{T_i} \varphi_i(z_1, \ldots, z_n) - \frac{1}{T_k} \varphi_k(z_1, \ldots, z_n). \qquad (4.2)$$

If the absolute value of the right-hand side of (4.2) is smaller than a sufficiently small positive number and is integrable with respect to t, then $z_i \approx z_k + \text{const}$. This relation holds for the angular coordinates of the rotors, and (for $\text{const} = 0$) for the voltages on the terminals of the generators of neighboring (i.e., with small mutual impedance) electric stations. Transformation of the initial equations on this basis is considered in /18/. Here, of course, it is also necessary to check local stability with respect to mutual oscillations of the machines for which the indicated relations are used—if there is no such stability, the right-hand side of (4.2) cannot be absolutely integrable. It can be easily shown that, mathematically, neglecting the differences between similar motions is also justified by the properties of systems of differential equations with sharply differing coefficients.

5. Integro-differential equations

A complex power system with sufficiently many units may be considered approximately as a system with distributed parameters /18/. To this end the numbers $\lambda_i = N_i (N_1 + \ldots + N_n)^{-1}$, $x_i = \lambda_1 + \ldots + \lambda_i$ are associated with the i-th unit (installed capacity N_i), and an independent variable x assuming the continuum of values $0 < x \leqslant 1$, including all the values x_i, is introduced. The corresponding generalized coordinates of the different units are considered not as a set of n functions of t, but as a single function of t and x. The equations of balance of the moments on the shafts of the units, the emf equations then take the form /18/:

$$T_a(x) \frac{\partial^2 \theta(t, x)}{\partial t^2} = S(t, x) - E(t, x) \int_0^1 \frac{E(t, y)}{Z(x, y)} \sin[\theta(t, x) - \theta(t, y) - \alpha(x, y)]\, dy, \qquad (5.1)$$

$$E'_d(t, x) = E(t, x) + \int_0^1 \frac{E(t, y)}{Y(x, y)} \cos[\theta(t, x) - \theta(t, y) - \alpha(x, y)]\, dy, \tag{5.2}$$

$$U^2(t, x) = \left\{E(t, x) - \int_0^1 \frac{E(t, y)}{X(x, y)} \cos[\theta(t, x) - \theta(t, y) - \alpha(x, y)]\, dy\right\}^2 + \\ + \left\{\int_0^1 \frac{E(t, y)}{X(x, y)} \sin[\theta(t, x) - \theta(t, y) - \alpha(x, y)]\, dy\right\}^2, \tag{5.3}$$

$$\int_{x_{k-1}}^{x_k} \int_{x_{l-1}}^{x_l} \frac{dx\, dy}{Z(x, y)} = \frac{\lambda_l \lambda_k}{Z_{lk}}, \quad \int_{x_{k-1}}^{x_k} \frac{dy}{Y(x, y)} = \lambda_l \frac{X_{q_l} - X'_{d_l}}{Z_{lk}}, \quad \int_{x_{k-1}}^{x_k} \frac{dy}{X(x, y)} = \lambda_k \frac{X_{q_l}}{Z_{l_k}}, \tag{5.4}$$

where T_a is the inertia constant of the rotor, related to the power of the unit, S is the mechanical moment, related to the nominal mechanical moment of the unit, and the remaining notations are the conventional ones. The remaining equations (of the prime movers, excitation systems of the generators, regulation of the units) are similarly transformed.

No restrictions other than (5.4) are imposed on the functions X, Y, and Z, i.e., these may be step functions. In this case the above integro-differential equations are only a new representation of the previous problem, permitting the application of new methods, which may turn out to be more effective than the old ones, to the given problem. If, however, X, Y, Z, and the other functions characterizing the parameters of the system (such as T_a) are well approximated by fairly smooth functions (which is apparently possible in most cases), then the unwieldy system of ordinary equations is approximated by a fairly compact integro-differential system.

In problems of static stability, i.e., stability under infinitesimal perturbations, this system becomes linear, and substitutions of the type $\Delta\theta = \varphi(x)e^{pt}$ reduce it to a system of homogeneous integral equations. Stability under infinitesimal perturbations is ensured if the real parts of all the eigenvalues p of the latter system are negative. Investigation of global stability, under a given perturbation, is possible by numerical solution of the system of integro-differential equations, which is a comparatively simple matter. A more profound analysis of global stability requires a generalization of Lyapunov's theory to continuous systems.

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F. R. Gantmakher and V.A. Yakubovich

ABSOLUTE STABILITY OF NONLINEAR CONTROL SYSTEMS

Introduction

We shall consider control systems containing one or several nonlinear units subjected, in general, to external forces. Such control systems are described by differential equations of the type

$$\left.\begin{aligned} &\frac{d\xi_j}{dt} = \sum_{h=1}^{\nu} \alpha_{jh}\xi_h + \sum_{k=1}^{\varkappa} \beta_{jk}\varphi_k(\sigma_k) + \psi_j(t),\\ &\sigma_k = \sum_{h=1}^{\nu} \gamma_{kh}\xi_h,\\ &(j, h = 1, \ldots, \nu, \quad k = 1, \ldots, \varkappa), \end{aligned}\right\} \tag{0.1}$$

where α_{jh}, β_{jk} and γ_{kh} are real constants, and without loss of generality $\varphi_k(0) = 0$.

Most of the paper will be devoted to a study of the absolute stability of the state $\xi_1 = \xi_2 = \ldots = \xi_\nu = 0$ in the absence of external action, i.e., when $\psi_j(t) \equiv 0$. To simplify the exposition, and since moreover the principal existing results refer to this case, we shall consider mainly the case when, in addition, the control system contains only one nonlinear unit ($\varkappa = 1$). In this case equations (0.1) become

$$\frac{d\xi_j}{dt} = \sum_{h=1}^{\nu} \alpha_{jh}\xi_h + \beta_j\varphi(\sigma), \quad \sigma = \sum_{h=0}^{\nu} \gamma_h\xi_h, \quad (j = 1, 2, \ldots \nu). \tag{0.2}$$

An extensive literature is concerned with systems (0.1); most of the results refer to the case $\nu = 2$. (The equations of Van der Pol, Liénard, and Duffing, for example, have the form of system (0.1) for the special case $\nu = 2$.) It is now clear that many of these results, originally obtained by various special methods (usually connected with qualitative investigations of phase curves in the plane), can also be obtained by a single method based on Lyapunov functions, and this method (this is particularly important) is applicable to systems (0.1) of arbitrarily high order ν.

The center of interest is thus shifted to effective methods for constructing Lyapunov functions or, rather, finding effectively verifiable conditions for their existence which guarantee their most important properties (absolute stability for $\psi(t) \equiv 0$, existence of limit motions for $\psi_j(t) \not\equiv 0$ which are stable in the large, etc.).

The present report is a survey of papers based on this approach. We shall not deal with papers studying particular cases of (0.1) by specific methods, or with approximate and asymptotic methods of importance for applications.

We proceed to describe briefly the history of the problem. We shall distinguish between the principal case, when the characteristic equation of the autonomous system*

$$\det\|a_{jh} - \delta_{jh}\lambda\| = 0 \tag{0.3}$$

has all its roots in the left half-plane, the simple critical case, when equation (0.3) has a single zero root and the other roots are in the left half-plane, and the general critical case, when all the roots of (0.3) are in the closed left half-plane, including an arbitrary number on the imaginary axis. The function $\varphi(\sigma)$ in (0.2) will at first be assumed to be continuous and to satisfy the condition

$$0 \leqslant \sigma\varphi(\sigma) \leqslant \mu_0\sigma^2 \qquad (\mu_0 \leqslant +\infty) \tag{0.4}$$

in the principal case and the condition

$$0 < \sigma\varphi(\sigma) \leqslant \mu_0\sigma^2 \qquad (\mu_0 \leqslant +\infty) \tag{0.5}$$

in the critical case. Condition (0.4) implies, obviously, that the graph of the function $\varphi(\sigma)$ is situated in the strip bounded by the straight lines $\varphi = 0$ and $\varphi = \mu_0\sigma$.

This approach was initiated by A.I. Lur'e /1/. He considers the problem of stability in the large of the trivial solution of the system (0.2) with $\mu_0 = \infty$ in the principal and simple critical cases. A Lyapunov function is sought in the form

$$V = \sum_{j,\,h=1}^{\nu} e_{jh}\xi_j\xi_h + \vartheta\int_0^{\sigma}\varphi(\sigma)\,d\sigma, \tag{0.6}$$

where $\vartheta = 0$ for the principal case and $\vartheta = 1$ for the simple critical case, and e_{jh} are specially selected coefficients depending on ν parameters. For these ν parameters a system of quadratic equations (Lur'e's resolvents) is obtained, the existence of a real solution of which guarantees stability in the large. For $\nu \leqslant 4$, conditions for the existence of solutions of these equations have been found /1/ in explicit form, thus effectively solving a large class of problems of practical importance.

Lur'e's method for systems (0.2) has been developed in different directions by A.M. Letov, P.V. Bromberg, V.M. Popov, Yu.I. Alimov, O.I. Komarnitskaya, A.K. Bedel'baev and others (see /2, 3/) and in its most general and final form by E.N. Rozenvasser /4/. Similar methods for constructing Lyapunov functions in the form (0.6) (or in an analogous form for the case of many nonlinearities) are considered by I.G. Malkin, N.N. Krasovskii, V.V. Rumyantsev, B.S. Razumikhin, E.A. Barbashin, V.A. Pliss, Yu.A. Alimov and others—see /2, 3/. In the USA a large group of scientists (G. Szegö, La Salle, R. Kalman, G. Bertram, G. Pearson, K. Meyer, and others — see /2, 3/) headed by S. Lefschetz is at present working in this direction.

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* In equation (0.3) $\delta_{jh} = 0$ for $j \neq h$, $\delta_{jj} = 1$.

Since Lur'e's method is based on the use of a Lyapunov function (0.6) of very special form (the Lyapunov function /4/ depends on $(\nu + 1)$ parameters, whereas the function (0.6) depends on $[\nu(\nu + 1)/2+1]$ parameters), it has long seemed that the conditions for absolute stability established by Lur'e's method can be considerably extended by the use of all Lyapunov functions of type (0.6).

In /5/ it was conjectured and proved for $\nu \leqslant 3$ that in the simple critical case the conditions obtained by Lur'e's method in fact include all conditions obtainable using Lyapunov functions of type (0.6). R. Kalman /6/ has proved an important algebraic proposition (/6/, "principal lemma"), which proves the conjecture for arbitrary ν and $\mu_0 = \infty$ for practically all systems (0.2) in the simple critical case (with the exception of certain systems corresponding to a set of measure zero in the space of the parameters $\alpha_{jk}, \beta_j, \gamma_j$). For the simple critical case and $\mu_0 \neq \infty$, a similar assertion is proved in /3/, using Kalman's result.* Quite recently a similar assertion for the principal and general critical cases has been proved /9/. All these results imply that in the problem of absolute stability of system (0.2) the class **L** of all functions of type (0.6) contains a certain optimal function V_0 with the property that if the class **L** contains at least one Lyapunov function for (0.2), V_0 is also a Lyapunov function. This optimal function V_0 is the very function constructed by a suitable application of Lur'e's method** (for the principal case, Rozenvasser's version of Lur'e's method /4/).

Another very useful approach to the problem of absolute stability of systems (0.2) has been developed by V.M. Popov /10—13/ (see also /3/). Without using Lyapunov functions, Popov derives (in the principal and simple critical cases) a certain frequency condition for absolute stability and proves that in the simple critical case this condition includes all the conditions obtainable using Lyapunov functions of class **L**. Popov proved a somewhat weaker assertion for the principal case as well (we mention that all absolute stability criteria obtained prior to Popov's results, with rare exceptions, use Lyapunov functions of class **L**, and are therefore included in Popov's criterion). An advantage of Popov's criterion, apart from its generality, is its extreme simplicity in application and its formulation in terms of the frequency response of the linear part of a system, a term familiar to control theoreticians.

Popov's method was later developed and applied by Ya. Z. Tsypkin, B.N. Naumov, A.Kh. Gelig, A. Halanay, C. Corduneanu and others (see /14—20/) to the proof of many new results.

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* To be specific, it was shown in /5, 7/ that in the simple critical case the existence problem for a Lyapunov function which is a quadratic form in the principal coordinates plus a nonlinearity integral (a somewhat narrower class than that of the functions (0.6)) leads to an algebraic problem — the solution of a special matrix inequality (see below, § 3, inequality (3.2)). The above-mentioned conjecture refers to this inequality (/5/, page 127). Since, as shown in /3/, the same inequality figures in more general problems of the construction of Lyapunov functions of type (0.6) (simple special case) or of the class of functions (0.6) given by an S-procedure /3/ (principal case), the conjecture /5/ implies that in all these cases Lur'e's method provides exhaustive conditions. The matrix inequality is solved in /8/ and the conjecture /5/ concerning it has been verified by Kalman /6/.

** Note that here by Lur'e's method we mean the method in which the "supplementary parameters" /5, 7/ are assumed equal to zero, or "limit" resolvents are considered /3/.

In /8/ the matrix inequalities /5/ arising in the search for necessary and sufficient (simple critical case) or sufficient, but in a certain sense optimal (principal case) conditions for the existence of Lyapunov functions of class **L** and a number of other problems, were solved. This result provides a new proof of Popov's criterion which implies that when Popov's condition is satisfied the system (0.2) has a Lyapunov function of class **L**. Thus, in the simple critical case Popov's condition is a necessary and sufficient condition for the existence of a Lyapunov function of class **L**.

Until recently it was not clear whether Popov's condition can be strengthened in the principal case by using a Lyapunov function of class **L**. Recently it has been proved /9/ that this is impossible: Popov's condition is also a necessary and sufficient condition in the principal case for the existence of a Lyapunov function of class **L**.*

Since, as mentioned above, Lur'e's method (for the principal case, in Rozenvasser's version) also includes all Lyapunov functions of class **L**, it also follows from /9/ that Popov's criterion is equivalent to the conditions obtained by Lur'e's method. This was proved before /9/, for the simple critical case and $\mu_0 = \infty$, by Kalman /6/, and for the simple critical case and $\mu_0 \neq \infty$ and for the principal case in the book /3/ (appendix). In this case substantial use is made of Kalman's result /6/.

Thus, by the efforts of a number of mathematicians the "Lur'e problem"—the problem of searching for effective necessary and sufficient existence conditions for Lyapunov functions of class **L** for the system (0.2)—has been solved. Much research has been devoted to this problem, starting with /21/, which first suggested using Lyapunov functions of class **L**. The conditions provided by Lur'e's method (for the principal case, in Rozenvasser's version /4/) and Popov's frequency condition turned out to be the required ones.

The results described above refer to the principal and simple critical cases. M.A. Aizerman and F.F. Gantmakher /22/ consider the general critical case under the assumption that the graph of the function $\varphi(\sigma)$ is separated from the critical line $\varphi = 0$, i.e., more accurately, that instead of (0.5) we have

$$\varepsilon\sigma^2 < \sigma\varphi(\sigma) \leqslant \mu_0\sigma^2 \tag{0.7}$$

for arbitrarily small $\varepsilon > 0$. It is shown that Popov's criterion carries over to this case, the parameter in his condition being determined as a rule from the condition that the frequency response passes near the critical values $\omega = \omega_0$ (the numbers $\pm i\omega_0$ are roots of equation (0.3)). Assuming (0.5), Popov's criterion is extended to the general critical case in /23—25/. In /25/ it is also shown that, as in the previously considered cases, Popov's criterion coincides with the conditions obtainable by Lur'e's method for that case, and is also a necessary and sufficient condition for the existence of a Lyapunov function of class **L**.

The results described above correspond, apart from a few details, to the contents of the book /3/ and §§ 2—5 of the present paper.** Recently a number of new results have been obtained, concerned with the study of systems (0.2) with hysteresis nonlinearities /26—28/, discontinuous nonlinearities /15, 25, 29/, and nonunique equilibrium position /18, 30/, and also systems (0.1) /16, 31/. Popov's condition has been strengthened /32, 33/ by using additional information about the nonlinearity (and expanding the class L of Lyapunov functions).

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* More accurately, at present some similar but nonequivalent formulations of Popov's frequency condition are known, which refer to similar, but different cases. The statement formulated above holds in all these cases; sometimes the Lyapunov function, while guaranteeing absolute stability, only satisfies the weak inequality $\dot{V} \leqslant 0$.

** The question asked in Chapter 5, § 4 of /3/ is also answered above.

Popov, Halanay /34/ and Gelig /18/ have extended Popov's criterion to the infinite-dimensional case (systems containing delay links, or described by partial differential equations). Tsypkin /14/, followed by Szego, Kalman, Pearson /35—37/ and others have obtained frequency conditions for absolute stability of sampled-data control systems. All these results are described in §§ 6—12.

In conclusion we make the following remark. At the present time the efforts of a number of authors have yielded many theorems of the Lyapunov type, which reduce some specific problem in the theory of differential equations to the existence problem for a function possessing some special properties, the most important of which is as a rule that its derivative should have a fixed sign everywhere or in some region determined by the given system of differential equations (the theorems of Yoshizawa /38—39/, Barbashin and Krasovskii /40/, Nemytskii /41—42/, Krasnosel'skii /43—44/, Zubov /45/, Lefschetz and La Salle /46/, Gelig /30/ and others, concerning the stability of the stationary state, stationary sets and manifolds, stability being understood in various senses, in the small and in the large: the existence of periodic and almost periodic motions, their stability and so on). In general it is almost impossible to indicate set methods for the construction of such functions. If, however, special systems are being considered, such as systems (0.1), and one is looking for existence conditions for the required function in some previously delineated class of functions (class L or the class of quadratic forms), then the problem usually reduces to a purely algebraic one of solving some special matrix inequality. In this connection different problems often lead to the same algebraic problems. Thus solution of these problems yields the solution of the initial problem in the form of effective, usually broad, sufficient conditions (they often turn out to be also necessary), since they comprise all the conditions obtainable by using Lyapunov functions of the class under consideration.

Some of these algebraic problems are considered below. Most of the results of the present paper can be obtained by the indicated method from the solution of these problems. However, many of the algebraic problems arising here remain unsolved.

We now proceed to a detailed exposition.

§ 1. Vector-matrix notations. Transfer functions and matrices. Definition of absolute stability.

In the sequel it will be convenient to use vector-matrix notation. This is due not so much to the need for abbreviation, as to the fact that the problems arising pertain essentially to matrix algebra and may be solved in that framework.

As usual, a rectangular matrix with ν_1 row and ν_2 columns is called a $\nu_1 \times \nu_2$ matrix. A matrix with $\nu_2 = 1$ (i.e., a column of numbers) is called a vector, and a matrix with $\nu_1 = 1$ (i.e., a row of numbers) is called a transposed vector. If A is a $\nu_1 \times \nu_2$ matrix, A^* denotes the transposed $\nu_2 \times \nu_1$ matrix with complex conjugate elements. In particular, if a, b are $\nu \times 1$ vectors, then a^*b is the scalar product of the vectors a and b, but ba^* is a $\nu \times \nu$ matrix. We shall denote by I the unit matrix, i.e., the matrix whose principal diagonal contains units and whose remaining elements are zeros.

Introducing the notation $x = \|\xi_j\|$, $P = \|\alpha_{jh}\|$, $q = \|\beta_{jk}\|$, $\varphi(\sigma) = \|\varphi_k(\sigma)\|$, $f(t) = \|\psi_j(t)\|$, $r = \|\gamma_{hk}\|$, $\sigma = \|\sigma_k\|$, $(j, h = 1, \ldots, \nu,\ k = 1, \ldots, \varkappa)$, we write system (0.1) in the form

$$\frac{dx}{dt} = Px + q\varphi(\sigma) + f(t), \qquad \sigma = r^*x, \tag{1.1}$$

where P is a $\nu \times \nu$ matrix, q and r are $\nu \times \varkappa$ matrices (and, consequently, r^* is a $\varkappa \times \nu$ matrix), x is a $\nu \times 1$ solution vector, $f(t)$ is a $\nu \times 1$ function vector (external action), σ, $\varphi(\sigma)$ are $\varkappa \times 1$ vectors.

System (0.2) can be written in a similar form

$$\frac{dx}{dt} = Px + q\varphi(\sigma), \qquad \sigma = r^*x, \tag{1.2}$$

where for system (0.2) $\varkappa = 1$, σ is a number, and $\varphi(\sigma)$ is a scalar function. We preserve the notation (1.2) also for an arbitrary $\varkappa > 1$.

Equation (0.3) takes the form

$$\det(P - \lambda I) = 0. \tag{1.3}$$

Consider system (0.1) or (1.1) with $\varkappa > 1$. Equating the external action to zero and separating nonlinear units, we obtain an autonomous linear control system (the linear part of system (1.1) or (1.2)) with inputs $(\varphi(t) = \|\varphi_j(t)\|$, outputs $\sigma(t) = \|\sigma_j(t)\|$ and equations $dx/dt = Px + q\varphi(t)$, $\sigma(t) = r^*x$. Introducing at the input $\varphi_1 = 0, \ldots, \varphi_{h-1} = 0, \varphi_h = e^{i\omega t}, \varphi_{h+1} = 0, \ldots, \varphi_\nu = 0$, we obtain at the output the values $\sigma_1 = -\chi_{1h}(i\omega)e^{i\omega t}, \ldots, \sigma_\nu = -\chi_{\nu h}(i\omega)e^{i\omega t}$. The values of $\chi_{jh}(i\omega)$ can be measured, the $\varkappa^2$ curves $\chi_{jh}(i\omega)$ for $0 \leqslant \omega < \infty$ are called, as is well-known, amplitude-phase response curves of the linear part of the system, the $\varkappa^2$ functions $\chi_{jh}(\lambda)$ are called transfer functions, and the $\varkappa \times \varkappa$ matrix

$$\chi(\lambda) = \|\chi_{jh}(\lambda)\| \tag{1.4}$$

is called the transfer matrix of the linear part of the system. For $\varkappa = 1$ we have one transfer function $\chi(\lambda)$ and one amplitude-phase response $\chi(i\omega)$. It is easily shown that for system (1.1) with arbitrary $\varkappa$

$$\chi(\lambda) = r^*(P - \lambda I)^{-1}q. \tag{1.5}$$

The transfer matrix $\chi(\lambda)$ is an invariant of the linear substitution $x = Sx_1$ in system (1.1), i.e., if substitution yields the system $dx_1/dt = P_1x_1 + q_1\varphi(\sigma_1) + f_1(t), \sigma = r_1^*x_1$ then $r^*(P - \lambda I)^{-1}q \equiv r_1^*(P_1 - \lambda I)q_1$. Other definitions of the transfer matrix $\chi(\lambda)$ are also known.

It is important to note that most of the results of the present paper are formulated in terms of the transfer matrix $\chi(\lambda)$ (or transfer function, if $\varkappa = 1$), i.e., these results can also be used in the frequently encountered case when the system of equations (1.1) or (1.2) is not given and only $\chi(\lambda)$

and certain general characteristics of the nonlinearities $\varphi_j(\sigma_j)$ and the external actions $\psi_j(t)$ are known.

Let us assume that some class M of nonlinearities $\varphi(\sigma) = \|\varphi_j(\sigma_j)\|$ is given, for example, let $\varphi_j(\sigma_j)$ be continuous functions satisfying the conditions

$$0 < \sigma_j\varphi_j(\sigma_j) \leqslant \mu_j\sigma_j^2 \tag{1.6}$$

for fixed numbers μ_j.

Our main interest is in conditions for absolute stability of the linear part of the system (1.2), i.e., conditions under which the solution $x \equiv 0$ of system (1.2) is asymptotically stable in the large for any $\varphi(\sigma) \in M$[*] We shall say that the linear part of system (1.2) or, briefly, system (1.2) is uniformly absolutely stable if

$$|x(t)| \leqslant \Theta(x_0, t), \tag{1.7}$$

where $\Theta(x_0, t)$ is some function depending only on the parameters of the linear part of the system and the numbers μ_j in formula (1.6), but not on $\varphi \in M$, such that $\Theta(x_0, t) \to 0$ uniformly in x_0 for $t \to \infty$, when x_0 varies in any bounded domain and $\Theta(x_0, t) \to 0$ for $|x_0| \to 0$ uniformly in t in every bounded subinterval of $[0, \infty)$.

These conditions have the following meaning. In a uniformly absolutely stable system the characteristics of interest (i.e., the bound (1.7)) do not change for arbitrary variation of the nonlinearities, as long as the latter remain in the class M. In an absolutely (but not uniformly) stable system, the order of decrease of an arbitrary solution may change when the nonlinearities vary, but it remains asymptotically stable in the large.

§ 2. Lur'e's method

First consider system (1.2) for the principal case and $\varkappa = 1$, assuming that $\varphi(\sigma)$ is a continuous function such that

$$\varphi(\sigma)\sigma > 0. \tag{2.1}$$

We require a Lyapunov function from the class L_0 of quadratic forms

$$V = x^*Hx, \qquad (H^* = H). \tag{2.2}$$

Since

$$\dot{V} = 2x^*H\dot{x} = -x^*Gx + 2x^*Hq\varphi,$$

where

$$G = -P^*H - HP, \tag{2.3}$$

[*] Asymptotic stability in the large means, as is well-known, that a) any solution $x(t) \to 0$ for $t \to \infty$; b) the solution $x(t) \equiv 0$ is asymptotically Lyapunov-stable.

we infer, using (2.1), that the required inequality $\dot{V} < 0$ for $x \neq 0$ will be satisfied if*

$$Hq = -r, \quad G > 0. \tag{2.4}$$

Let us try to find a matrix H satisfying these conditions by taking $G = uu^* + \varepsilon I$, where u is some vector and $\varepsilon > 0$. It is well-known that under our assumptions regarding the roots of equation (1.3) the solution H of the matrix equation (2.3) (or, in scalar form, of the corresponding system of $\nu(\nu + 1)/2$ linear nonhomogeneous equations) is uniquely determined, and $H > 0$, if $G > 0$. Let us denote this solution by

$$H = L(G). \tag{2.5}$$

In our case, by virtue of the linearity of the operator $L(G)$,

$$H = L(uu^*) + \varepsilon L(I). \tag{2.6}$$

Substituting in (2.4), we obtain

$$L(uu^*)q + r' = 0, \tag{2.7}$$

where

$$r' = r + \varepsilon L(I). \tag{2.8}$$

The vector equation (2.7) in scalar notation is a system of ν quadratic equations for the components of the vector u—Lur'e's resolvents. If these equations have a real solution u for all vectors r' sufficiently close to r, there is, in particular, a solution u for the vector r' defined by formula (2.8) with sufficiently small $\varepsilon > 0$. Determining the matrix H by formula (2.6), we find that (2.3) and (2.4) are satisfied with $G = uu^* + \varepsilon I$. Since $G > 0$, there exists $\delta > 0$ such that $x^*Gx \geqslant 2\delta x^*Hx$, for all x. Then $\dot{V} = -x^*Gx - \sigma\varphi(\sigma) \leqslant -2\delta V$, i.e., $V[x(t)] \leqslant e^{-2\delta t}V[x(0)]$. This means that for some constant $\Upsilon > 0$, depending only on H (i.e., on P, q, and r), but not on the function $\varphi(\sigma)$, we have

$$|x(t)| \leqslant \gamma e^{-\delta t}|x(0)|. \tag{2.9}$$

The argument remains valid when $\varphi = \varphi(\sigma)$ is an arbitrary function depending on x, t and, possibly (provided (2.1) is satisfied), on certain parameters. The function $\varphi(\sigma)$ may be, in particular, a hysteresis function.

Let us formulate our result.

Theorem 2.1. *Assume that system (1.2), where $\varkappa = 1$, corresponds to the principal case (all the roots of equation (1.3) lie in the left half-plane), and the output of the nonlinear unit $\varphi = \varphi(\sigma)$ is an arbitrary function of time related to the input σ by (2.1). Assume that for all vectors r' sufficiently*

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* Here and in the sequel the inequality $G > 0$ means that G is a positive-definite matrix, i.e., $x^*Gx > 0$ for any vector $x \neq 0$. Similarly, $G \geqslant 0$ implies that $x^*Gx \geqslant 0$ for any x.

close to r the quadratic vector equation (2.7) has a real solution u, where the operator $L(G)$ is defined by (2.3) and (2.5). Then system (1.2) is uniformly absolutely stable, and, moreover, there exist constants $\gamma > 0$, $\delta > 0$, depending on the parameters of the linear part of system (1.2), such that for any solution $x(t)$ inequality (2.9) holds.

Again, let $\varphi(\sigma)$ be a continuous function, and instead of (2.1) suppose

$$0 \leqslant \sigma\varphi(\sigma) \leqslant \mu_0\sigma^2, \qquad (\mu_0 \leqslant +\infty). \tag{2.10}$$

We require a Lyapunov function of the class **L** of functions of the form

$$V = x^*Hx + \vartheta\int_0^\sigma \varphi(\sigma)\,d\sigma, \tag{2.11}$$

where the matrix H and the number ϑ are the required parameters. After elementary calculations we obtain

$$-\dot V = x^*Gx + 2x^*g\varphi + \gamma\varphi^2 + (\sigma - \varphi\mu_0^{-1})\varphi, \tag{2.12}$$

where G is defined by (2.3) and

$$-g = Hq + \tfrac{1}{2}(\vartheta P^* + I)r, \quad \gamma = \mu_0^{-1} - \vartheta r^*q. \tag{2.13}$$

For $\gamma > 0$, $\dot V$ can be written in the form

$$-\dot V = x^*(G - \gamma^{-1}gg^*)x + \gamma|\varphi + \gamma^{-1}g^*x|^2 + (\sigma - \varphi\mu_0^{-1})\varphi.$$

Since (2.10) implies that $(\sigma - \varphi\mu_0^{-1})\varphi > 0$, $\dot V < 0$ will hold for $x \neq 0$, if H and ϑ satisfy

$$\gamma > 0, \quad \gamma G - gg^* > 0, \tag{2.14}$$

or

$$\gamma = 0, \quad g = 0, \quad G > 0. \tag{2.15}$$

The second inequality of (2.14) will be satisfied if we take $g = \sqrt{\gamma}\,u$, $G = uu^* + \varepsilon I$, where $\varepsilon > 0$ is arbitrarily small. Then H is found from (2.6): as above, we obtain the resolvent

$$\mathbf{L}(uu^*)q + \sqrt{\gamma}u + \tfrac{1}{2}(\vartheta P^* + I)r' = 0, \tag{2.16}$$

which must have a solution for all vectors r' sufficiently close to r. In addition, the parameter ϑ must satisfy $\gamma = \mu_0^{-1} - \vartheta r^*q > 0$, $\det(\vartheta P^* + I) \neq 0$. Relations (2.15) lead, as above, to equation (2.16) with $\gamma = 0$.

If equation (2.16) has a real solution u for all vectors r' sufficiently close to the given vector r and for some parameter ϑ, then $\dot V < 0$ for $x \neq 0$; and hence, as is easily shown, follows the uniform absolute stability in the class of continuous nonlinearities satisfying condition (2.10).[*] Equation (2.16) (in a somewhat different form) was derived by Rozenvasser /4/. Equation (2.7) results from (2.16) by putting $\mu_0 = \infty$, and $\vartheta = 0$.

We supplement the above results by a few remarks. One could take $\varepsilon = 0$ in the above argument. Then, instead of the "varied" Lur'e resolvents (2.7) and (2.16), "unvaried" equations with $r' = r$ would be obtained. However, the existence of a real solution of these equations ensures only the inequality $\dot{V} \leqslant 0$, and therefore stability in the large may be deduced only under additional assumptions.

As indicated above, the solvability conditions of equations (2.7) and (2.16) for $\nu \leqslant 4$ (and fixed ϑ) are expressed in terms of the coefficients of these equations in /1, 2, 5/. These conditions can also be found as follows. We determine polynomials $\alpha(\lambda)$ and $\Delta(\lambda)$ from the relation $\mu_0^{-1} + (1 + \lambda\vartheta)\chi(\lambda) = \alpha(\lambda)/\Delta(\lambda)$, where, we recall, $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$.

Therefore, $\Delta(\lambda) = \det(P - \lambda I)$. *In order that equation (2.16), where $\gamma > 0$, have a real solution u for all r' sufficiently close to r, it is necessary and sufficient that there exist a polynomial $\xi(\lambda)$ with real coefficients which does not have purely imaginary zeros and which satisfies the identity*

$$\alpha(\lambda)\Delta(-\lambda) + \alpha(-\lambda)\Delta(\lambda) \equiv \xi(\lambda)\xi(-\lambda). \tag{2.17}$$

For $\gamma = 0$ the additional condition $\lim_{\omega\to\infty} \omega^2\alpha(i\omega/\Delta(i\omega) > 0$ must hold.

Equations (2.16) and (2.17) remain valid in the general critical case as well, if certain additional conditions are satisfied /25/.

In concluding this section we note that the above version of Lur'e's method may be carried over almost unchanged to the case of many nonlinearities, i.e., $\varkappa > 1$.

The proof of Theorem 2.1 remains the same word for word, except that u is a $\nu \times \varkappa$ matrix. The matrix equation (2.7), in scalar notation, will be a system of $\nu\varkappa$ quadratic equations for the elements of the matrix u.

For $\varkappa > 1$ one should consider, instead of (2.11), a function

$$V = x^*Hx + \sum_{j=1}^{\varkappa} \theta \int_0^{\sigma_j} \varphi_j(\sigma_j)\,d\sigma_j. \tag{2.18}$$

Then, instead of (2.16), we obtain the equation (taking $g = u$)

$$L(u\gamma^{-1}u^*)q + u + \frac{1}{2}(P^*r'\vartheta_d + r'\tau_d) = 0, \quad \gamma = \tau_d\mu_d^{-1} - \vartheta_d r^*q, \tag{2.19}$$

where τ_d and ϑ_d are diagonal matrices with diagonal elements to be chosen respectively, $\tau_1 > 0, \ldots, \tau_\varkappa > 0$, $\vartheta_1, \ldots, \vartheta_\varkappa$, and μ_d is a diagonal $\varkappa \times \tilde{\varkappa}$ matrix with diagonal elements $\mu_1, \ldots, \mu_\varkappa$. Conditions for solvability of equations (2.19) are known, however, only in a few particular cases.

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[*] Now, however, this conclusion is not necessarily true for nonlinearities of a more general class than that of continuous functions satisfying (2.10). In this connection, the narrower class L_0 of Lyapunov functions (2.1) has some advantage over the class L of Lyapunov functions (2.11); it is useful to bear this in mind in the sequel.

§ 3. Special matrix inequalities and their solution

The problem of constructing a Lyapunov function of class $\mathbf{L}_0$ (see (2.2)) or $\mathbf{L}$ (see (2.11)) was reduced in § 2 to the following algebraic problems, these are more conveniently formulated in somewhat different notations.

Given: a real Hurwitz $\nu \times \nu$ matrix A (the equation $(A - \lambda I) = 0$ has all its roots in the open left half plane), $\nu \times 1$ real vectors a, b and a number $\rho > 0$.* Using a matrix $H = H$ we define

$$G = -(A^*H + HA), \quad g = -(Ha + b). \tag{3.1}$$

It is required to indicate conditions for the existence of a real symmetric matrix H such that either

$$\rho G - gg^* > 0, \tag{3.2}$$

or

$$G > 0, \quad g = 0, \tag{3.3}$$

or

$$Q \equiv \begin{pmatrix} G & g \\ g^* & \rho \end{pmatrix} > 0, \tag{3.4}$$

where in (3.4) we may have $\rho = 0$.**

We note that in cases (3.2) and (3.3) $G > 0$ and therefore, necessarily, $H > 0$, in accordance with Lyapunov's well-known theorem.

In the case (3.4), $G \geq 0$ and, consequently, $H \geq 0$. Since the Lyapunov function must be positive-definite, in this latter case we also need conditions for the inequality $H > 0$ to hold.

The solution of these algebraic problems is given by the following theorems.

Theorem 3.1. *For the existence of a solution $H = H^*$ of inequality (3.2) it is necessary and sufficient that for all ω in the interval $0 \leq \omega < \infty$:*

$$\rho + 2\,\mathrm{Re}\, b^*(A - i\omega I)^{-1} a > 0. \tag{3.5}$$

Theorem 3.2. *For the existence of a matrix $H = H^*$, satisfying (3.3), it is necessary and sufficient that:*

$$\mathrm{Re}\, b^*(A - i\omega I)^{-1} a > 0 \text{ for } 0 \leq \omega < \infty, \tag{3.6}$$

$$\lim_{\omega \to \infty} \omega^2\, \mathrm{Re}\, b^*(A - i\omega I)^{-1} a > 0. \tag{3.7}$$

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* The complex case is considered analogously.

** The problem of constructing a Lyapunov function of class $\mathbf{L}_0$ ($\rho = 0$ or $\mathbf{L}$ ($\rho > 0$), such that $\dot{V} \leq 0$, leads to inequality (3.4).

Theorem 3.3. *a) For the existence of a matrix $H = H^*$ satisfying (3.4), it is necessary that*

$$\rho + 2\,\mathrm{Re}\, b^*(A - i\omega I)^{-1} a > 0, \text{ for } 0 \leqslant \omega < \infty. \tag{3.8}$$

b) If the rank of the matrix

$$\| a, Aa, \ldots, A^{\nu-1}a \| \tag{3.9}$$

is ν, then condition (3.8) is also sufficient. c) The null-space of the matrix H is contained in a subspace which is orthogonal to that spanned by the column vectors of the matrix

$$\| b, A^*b, \ldots A^{*\nu-1}b \|. \tag{3.10}$$

If, in particular, the rank of the matrix (3.10) is ν, then $H > 0$.

d) If (3.8) is satisfied and the rank of the matrix (3.9) is ν, then there exists a matrix $H = H^*$ such that

$$G = uu^*, \quad g = u\sqrt{\rho}, \tag{3.11}$$

where u is some vector.

We note that (3.11) obviously implies inequality (3.4). For $\rho > 0$, relations (3.11) can be written in the form $\rho G - gg^* = 0$, which indicates their relation to condition (3.2).

Theorems 3.1 and 3.2 are proved in /8/. The proof of part a) of Theorem 3.3 is a word for word repetition of the corresponding reasoning in /8/. Parts b) and c) can be derived from Theorem 3.1 by passing to the limit. A proof of parts b), c) and d) of Theorem 3.3, independent of /8/, has been given by Kalman /6/.

We give a proof of Theorem 3.2 based on the arguments in /6, 8/; to simplify matters we assume that the rank of the matrix (3.9) is ν. We denote $A_\omega = A - i\omega I$, $a_\omega = (A - i\omega I)^{-1}a$. Assuming the required matrix H to exist, we have (ω being real)– $G = A_\omega^* H + HA_\omega$. Multiplying this relation on the left by a_ω^* and on the right by a_ω and substituting $Ha = -b$ (which follows from $g = 0$), we obtain

$$2\mathrm{Re}\, b^*(A - i\omega I)^{-1}a \equiv a_\omega^* G a_\omega. \tag{3.12}$$

Hence

$$b^*Aa = \lim_{\omega\to\infty} \omega^2\, \mathrm{Re}\, b^*(A - i\omega I)^{-1} a = \frac{1}{2} a^* G a. \tag{3.13}$$

Formulas (3.12) and (3.13) imply the necessity of conditions (3.6) and (3.7).

Now suppose (3.6) and (3.7) are satisfied. We show that a matrix $H = H^*$ exists which satisfies (3.1) and (3.3).

We show first that (3.3) follows from (3.12). Substituting $G = -(A_\omega^* H + HA_\omega)$ in (3.12) we obtain

$$\mathrm{Re}\,(Ha + b)^*(A - i\omega I)^{-1} a \equiv 0. \tag{3.14}$$

The function $\psi(\lambda) = (Ha + b)^* (A - \lambda I)^{-1} a$ is holomorphic in the left half-plane, and $\operatorname{Re} \psi(i\omega) \equiv 0$. By the principle of analytic continuation $\psi(\bar{\lambda}) = -\overline{\psi(-\lambda)}$, i.e., $\psi(\lambda)$ is holomorphic everywhere, and consequently $\psi(\lambda) \equiv \text{const}$. Asymptotic considerations for $|\lambda| \to \infty$ imply that $\psi(\lambda) \equiv 0$, and by virtue of the linear independence of the columns of the matrix (3.9), $g = -(Ha + b) = 0$. Thus, it is sufficient to find a matrix $G > 0$ satisfying (3.12) (then H will be determined from the first relation of (3.1)). We will look for G in the form $G = uu^* + \varepsilon I$, where $\varepsilon > 0$. Substituting in (3.12) we obtain we obtain

$$2\operatorname{Re} b^* a_\omega - \varepsilon a_\omega^* a \equiv |u^* a_\omega|^2. \tag{3.15}$$

From (3.6) and (3.7) it follows that for sufficiently small $\varepsilon > 0$ the left-hand side of (3.15) is positive for all $\omega > 0$. Let $\Delta(\lambda) = \det(A - \lambda I)$, $(A - \lambda I)^{-1} a = \frac{q(\lambda)}{\Delta(\lambda)}$, $q(\lambda) = q_0 + \lambda q_1 + \ldots + \lambda^{\nu-1} q_{\nu-1}$. The condition on the matrix (3.9) implies the linear independence of the vectors $q_0, q_1, \ldots, q_{\nu-1}$. The equality (3.15) can be written in the form $\beta(i\omega) = |u^* q(i\omega)|^2$,

$$\beta(i\omega) = |u^* q(i\omega)|^2, \tag{3.16}$$

where $\beta(\lambda)$ is a polynomial in λ^2 with real coefficients, defined by the relation $\beta(i\omega) = |\Delta(i\omega)|^2 \cdot [2\operatorname{Re} b^* a_\omega - \varepsilon a_\omega^* a_\omega]$. Since $\beta(i\omega) > 0$ for $\omega > 0$, the roots of the equation $\beta(\lambda) = 0$ are symmetrical with respect to both the real axis and the imaginary axis. Therefore there exists a polynomial $\xi(\lambda)$ with real coefficients which satisfies the identity

$$\beta(\lambda) \equiv \xi(\lambda) \xi(-\lambda). \tag{3.17}$$

For $\xi(\lambda)$ we take a product, with suitable coefficient, of the monomials $(\lambda - \lambda_j)$, where λ_j are the roots of the equation $\beta(\lambda) = 0$ in the left half-plane. The resulting polynomial $\xi(\lambda) = \xi_0 + \lambda \xi_1 + \ldots + \lambda^{\nu-1} \xi_{\nu-1}$ is used to determine the vector u from the relation $\xi(\lambda) = u^* q(\lambda)$, i.e., from the relations $\xi_j = u^* q_j$ $(j = 0, 1, \ldots, \nu - 1)$. This is possible by virtue of the linear independence of the vectors q_j. When this is done (3.16), and consequently (3.15) and (3.11) are satisfied, which completes the proof.

Theorems 3.1 and 3.3 can be proved similarly.

In the case of many nonlinearities, problems of constructing Lyapunov functions of the form $V = x^* H x + \sum_{j=1}^{\varkappa} \vartheta_j \int_0^{\sigma_j} \varphi_j \, d\sigma_j$ lead to analogous matrix inequalities, in which now a and b are $\nu \times \varkappa$ matrices, and $\rho > 0$ is a $\varkappa \times \varkappa$ symmetric matrix. Instead of (3.2) one considers the inequality

$$G - g \rho^{-1} g^* > 0. \tag{3.18}$$

Let us introduce the notation $\operatorname{Re} Z = \frac{1}{2}(Z + Z^*)$, where Z is an arbitrary complex matrix. The following assertion can be proved.

Theorem 3.4. *Using the above notation, the following remain valid: Theorem 3.1 with (3.2) replaced by (3.8), Theorem 3.2 with the condition that rank* $a = \varkappa$, *and Theorem 3.3, where* u *is a matrix of order* $\nu \times \varkappa$; *in (3.5), (3.6) and (3.8)* $-\infty < \omega < +\infty$.

In this case the left-hand sides of inequalities (3.5)—(3.8) are $\varkappa\times\varkappa$ matrices, and the matrices (3.9) and (3.10) are of order $\nu\times\nu\varkappa$.

The problem of necessary and sufficient conditions for the existence of a Lyapunov function of class L for system (1.2) with $\varkappa=1$, where $\varphi(\sigma)$ satisfies conditions (0.3), leads to the following matrix problem.

Given A, a, b, c, γ, where A is a $\nu\times\nu$ real Hurwitz matrix, a, b, c are real $\nu\times 1$ vectors and γ is a real number, we determine g and G by formulas (3.1) and set $F(\mu, H)=G+\mu(gc^*+cg^*)+\gamma\mu^2cc^*$. It is required to find existence conditions for a matrix $H=H^*$ such that

$$F(\mu, H)>0 \text{ for } 0\leqslant\mu\leqslant\mu_0, \tag{3.19}$$

where μ_0 is a given number (for the problem indicated above $A=P$, $a=q$, $b=\frac{1}{2}\vartheta Pr^*$, $c=r$, $\gamma=-\vartheta r^*q$.)

Theorem 3.5. *For the existence of a matrix $H=H^*$ satisfying (3.19), it is necessary and sufficient that for some $\tau>0$ and all $\omega>0$*

$$\gamma+2\operatorname{Re}b^*(A-i\omega I)^{-1}a+\tau[\mu_0^{-1}+\operatorname{Re}c^*(A-i\omega I)^{-1}a]>0.$$

The proof is given in /9/, where the case of the weak inequality $F(\mu, H)\geqslant 0$ is also considered.

It is important to note that the problems considered above of the necessary and sufficient conditions for the existence of Lyapunov functions of type (2.1) and (2.11) in the problem of the absolute stability of systems (1.2) are not the only ones leading to the matrix inequalities (3.2)—(3.4), (3.19); the same is true of various other problems considered below for systems (1.1) and (1.2).

§ 4. Popov's frequency condition

Consider system (1.2) for $\varkappa=1$, where $\varphi(\sigma)$ is a continuous function satisfying relations (0.4). Applying Theorems 3.1 and 3.2 to the matrix inequalities (2.14) and (2.15), we obtain the following sufficient (and necessary) conditions for the existence of a solution $H=H^*$:

$$\pi(\omega)\equiv\mu_0^{-1}+\operatorname{Re}[(1+i\omega\vartheta)\chi(i\omega)]>0 \text{ for } 0\leqslant\omega<\infty, \tag{4.1}$$

$$+\infty>\lim_{\omega\to\infty}\omega^2\pi(\omega)>0. \tag{4.2}$$

If (4.1) holds, inequality (4.2) can be satisfied, as shown in /3/ (pp. 85-86), by a sufficiently small variation of the parameter ϑ. We arrive at the following result (for more details see /26, 47/):

Theorem 4.1. (Popov's criterion). *In system (1.2), let P be a Hurwitz matrix and let (4.1) be satisfied for some parameter ϑ. Then the linear part of system (1.2) is uniformly absolutely stable for the class of continuous functions $\varphi(\sigma)$ satisfying (0.4).*

As mentioned in the introduction, this theorem* was obtained by Popov /12/ in a different way. Let us describe his method, assuming for simplicity that instead of (0.4) we have the stronger conditions

$$|\varphi(\sigma)| \leqslant \varphi_{\lim}, \quad 0 < \sigma\varphi(\sigma) < \mu_0\sigma^2 \quad (\mu_0 \leqslant +\infty). \tag{4.3}$$

On the basis of this condition we may assume that instead of (4.1) we have the somewhat weaker condition

$$\mu_0^{-1} + \mathrm{Re}\,[(1 + i\omega\vartheta)\chi(i\omega)] > 0 \text{ for } 0 \leqslant \omega < \infty. \tag{4.4}$$

We shall show that under these assumptions any solution of system (1.2) tends to zero for $t \to \infty$. We give the proof for $\vartheta > 0$ (the case $\vartheta < 0$ is analogous but slightly more complicated).

From (1.2) we have

$$x(t) = e^{Pt}x(0) + \int_0^t e^{P(t-\tau)} q\varphi[\sigma(\tau)]\,d\tau, \tag{4.5}$$

$$\sigma(t) = \sigma_0(t) + \int_0^t \Omega(t-\tau)\,\varphi[\sigma(\tau)]\,d\tau, \tag{4.6}$$

where $\sigma_0(t) = r^* e^{Pt} x(0)$, $\Omega(t) = r^* e^{Pt} q$.

There exist $\gamma > 0$ and $\varepsilon > 0$ such that $|e^{Pt}| \leqslant \gamma e^{-\varepsilon t}$. From (4.5) it follows that

$$|x(t)| \leqslant \gamma e^{-\varepsilon t}|x(0)| + \int_0^t \gamma e^{-\varepsilon(t-\tau)}|q|\,|\varphi(\sigma)|\,d\tau, \tag{4.7}$$

$$|x(t)| \leqslant \gamma e^{-\varepsilon t}|x(0)| + \gamma|q|\varphi_{\lim}/\varepsilon, \tag{4.8}$$

i.e., $x(t)$ and $\sigma(t) = r^* x(t)$ are bounded functions of time. We shall show that $\sigma(t) \to 0$ for $t \to \infty$. Then $\varphi[\sigma(t)] \to 0$ ($\varphi(\sigma)$ is a continuous function) and from (4.7) it will follow by l'Hôpital's rule that $|x(t)| \to 0$ for $t \to \infty$.

Let us denote

$$\varphi_T(t) = \varphi[\sigma(t)] \text{ for } 0 \leqslant t \leqslant T, \; \varphi_T = 0 \text{ for } t > T \text{ or } t < 0, \tag{4.9}$$

$$\xi(t) = \int_0^t \Omega(t-\tau)\,\varphi_T(\tau)\,d\tau. \tag{4.10}$$

Here $T > 0$ is for the time being arbitrary. Consider the integral

$$J_T = \int_{-\infty}^{+\infty} [(\xi - \varphi_T\mu_0^{-1})\varphi_T + \vartheta\dot{\xi}\varphi_T]\,d\tau. \tag{4.11}$$

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* To be precise, a similar but somewhat weaker theorem. Condition (4.1) can be weakened (replacing $>$ by $\geqslant$) by imposing certain additional restrictions.

By Parseval's equation

$$J_T = \int_{-\infty}^{+\infty} [(\tilde{\xi} - \tilde{\varphi}_T \mu_0^{-1}) \tilde{\varphi}_T^* + \vartheta (i\omega) \tilde{\xi} \tilde{\varphi}_T^*] \, d\omega, \tag{4.12}$$

where $\tilde{\xi}$, $\tilde{\varphi}_T$ are the Fourier transforms of the functions ξ, φ_T, and the star denotes the complex conjugate. It follows from (4.10) that $\tilde{\xi} = -\chi \sim (i\omega) \tilde{\varphi}_T$, where $\chi(\lambda)$ is the transfer function (1.5). Substituting the value of $\tilde{\xi}$ in (4.12) we obtain (using the fact that J_T is real)

$$J_T = -\int_{-\infty}^{+\infty} \operatorname{Re} [\mu_0^{-1} + (1 + i\omega\vartheta) \chi(i\omega)] \cdot |\tilde{\varphi}_T|^2 \, d\omega. \tag{4.13}$$

It follows from (4.14) that $J_T \leqslant 0$. Substituting $\xi(t) = \sigma(t) - \sigma_0(t)$ in (4.11) and replacing $\int_{-\infty}^{+\infty}$ by $\int_0^T$ and φ_T by φ, (which is permissible by virtue of (4.19)), we obtain

$$J_T = \int_0^T [(\sigma - \varphi\mu_0^{-1})\varphi + \vartheta\dot{\sigma}\varphi + (\sigma_0 + \vartheta\dot{\sigma}_0)\varphi] \, dt. \tag{4.14}$$

From (4.3) we conclude that $\psi(\sigma) \equiv [\sigma - \varphi(\sigma)\mu_0^{-1}]\varphi(\sigma) > 0$ for $\sigma \neq 0$. Since $J_T \leqslant 0$, $\vartheta > 0$, it follows from (4.14) that

$$\int_0^T \psi[\sigma(t)] \, dt \leqslant \int_0^\infty |\sigma_0 + \vartheta\dot{\sigma}_0| \, dt \cdot \varphi_{\text{lim}} + \vartheta \int_0^{\sigma(0)} \varphi \, d\sigma \tag{4.15}$$

and since $T > 0$ is arbitrary, the integral $\int_0^\infty \psi[\sigma(t)] \, dt$ converges. The function $\dot{\sigma}(t) = r^*(Px + q\varphi)$ is bounded. If we had $\sigma(t) \nrightarrow 0$ for $t \to \infty$, then for certain $t_n \to \infty$ and $\delta > 0$ we should have $|\sigma(t_n)| > 2\delta$, and consequently also $|\sigma(t)| < \delta$ in the intervals $(t_n - \varepsilon, t_n + \varepsilon)$, where $\varepsilon > 0$ is some number determined by $\sup|\dot{\sigma}|$. Then $\int_0^\infty \psi[\sigma(t)] dt = \infty$. This contradiction shows that $\sigma(t) \to 0$ for $t \to \infty$, and the proof is complete.

We note that the estimate (4.15) easily yields absolute stability in the class of nonlinearities satisfying (4.3).

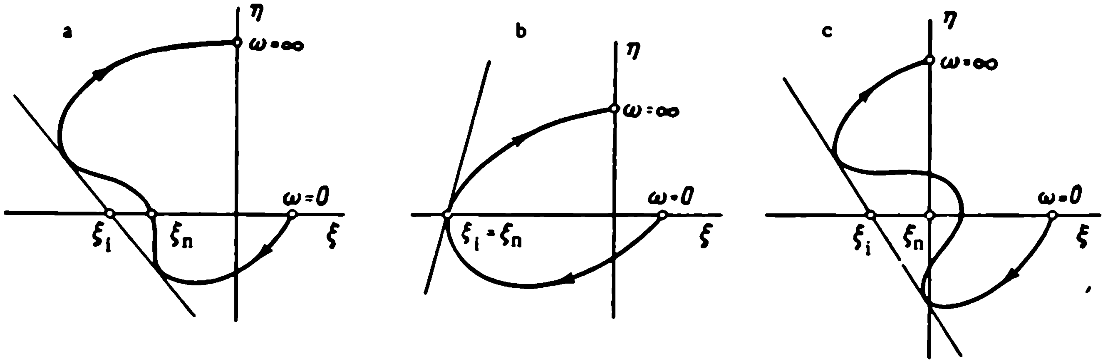
The frequency conditions (4.1) or (4.4) have a simple geometrical interpretation. Put

$$\xi(\omega) = \operatorname{Re} \chi(i\omega), \quad \eta(\omega) = \omega \operatorname{Im} \chi(i\omega) \tag{4.16}$$

and construct in the $\{\xi, \eta\}$ plane the curve $\xi = \xi(\omega)$, $\eta = \eta(\omega)$, which, following Popov, we shall call the modified frequency response. The curve $\xi = \xi(\omega)$, $\eta = \eta(\omega)$ is obviously obtained from the frequency response $\chi(i\omega)$ by multiplying the ordinates by ω. From (1.5) it follows that

$$\xi(\omega) = r^*(P^2 + \omega^2 I)^{-1} Pq, \quad \eta(\omega) = \omega^2 r^*(P^2 + \omega^2 I)^{-1} q. \qquad (4.17)$$

In the $\{\xi, \eta\}$ plane draw a straight line (Popov's line) such that: a) the modified frequency response is to the right of this line, b) the abscissa ξ_p of the point of intersection of this line with the axis $\eta = 0$ is nonpositive, c) the value of ξ_p is the maximum possible (Popov's line may obviously contain points in common with the modified frequency response (see Figure 1)). We set $\mu_i = -\xi_i^{-1}$ if $\xi_i \neq 0$, and $\mu_i = \infty$ if $\xi_i = 0$. Popov's criterion (condition (4.1)) states that there is absolute stability in the class of nonlinearities satisfying $0 \leqslant \varphi(\sigma)/\sigma \leqslant \mu_0 < \mu_i$, if $\mu_i \neq \infty$, or $\varphi(\sigma)/\sigma > 0$, if $\mu_i = \infty$. Condition (4.4) is interpreted analogously.



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The connection between the Popov and Nyquist criteria is easily established. We denote by ξ_n the abscissa of the extreme left point of intersection of the modified response with the axis $\eta = 0$ if this abscissa is nonpositive, otherwise $\xi_n = 0$ (see Figure 1). The value of ξ_n is obviously unchanged if the modified response is replaced by the frequency response $\chi(i\omega)$. Therefore, according to the Nyquist criterion, the linear system obtained from (1.2) for $\varphi(\sigma) = \mu\sigma$ will be stable for $0 < \mu < \mu_n = \xi_n^{-1}$, if $\xi_n \neq 0$, or for all $\mu > 0$, if $\xi_n = 0$. It may happen that $\xi_n = \xi_i$ (see Figure 1b). Aizerman's well-known problem is relevant for systems of this type.

Popov's criterion is thus very simple to apply and very intuitive.

We mention another important fact. The above proof of condition (4.4), which does not use the matrix inequalities of § 3, remains unchanged when the system is described by equation (4.6), where the function $\Omega(t)$ is an arbitrary continuously differentiable function satisfying the condition

$$|\Omega(t)| + |\dot{\Omega}(t)| \leqslant \gamma e^{-\varepsilon t}, \qquad (4.18)$$

where $\gamma > 0$, $\varepsilon > 0$, or some other less restrictive condition. The above assertions are made more precise by the following theorem /18/.

Theorem 4.2. *Suppose the transfer function* $\chi(p)$ *is analytic in the half-plane* $\operatorname{Re} p > -\varepsilon$, *where* $\varepsilon > 0$, *and* $\sigma_0(t)$, $\Omega(t)$ *are continuously differentiable and satisfy inequality (4.18), and* $|\sigma_0(t)| + |\dot{\sigma}_0(t)| \leqslant \gamma_0 e^{-\varepsilon t}$.

Then if Popov's condition (4.1) is satisfied, the system described by equation (4.6) is absolutely stable for the class of continuous nonlinearities $\varphi(\sigma)$ *satisfying* $0<\sigma\varphi(\sigma)<\mu_0\sigma^2$ $(\mu_0\leqslant\infty)$, *in the sense that* $\lim\limits_{t\to\infty}\sigma(t)=0$ *and* $\sup\limits_{t>0}|\sigma(t)|\to 0$ *for* $\gamma_0\to 0$.

Theorem (4.2) may be applied to systems more general than (1.2), for example, to systems with a delay of the form

$$\frac{dx}{dt}=\sum_{j=1}^{h}P_jx(t-\tau_j)+q\varphi(\sigma),\ \sigma=r^*x, \tag{4.19}$$

where $\tau_j>0$, and others. In such cases it is sufficient to know only the frequency response $\chi(i\omega)$ of the linear part of the system. In the case of system (4.19) we obviously have

$$\chi(\lambda)=r^*\Big(\sum_{j=1}^{h}P_je^{-\tau_j\lambda}-\lambda I\Big)q.$$

Theorems 4.1 and 4.2 may also be extended to the case of many nonlinearities. The following generalization of Theorem (4.1) is valid in this case.

Theorem 4.3. (Popov's criterion for the case of many nonlinearities). *Consider system (1.2) for* $\varkappa>1$, *where* P *is a Hurwitz matrix, and the continuous functions* $\varphi_j(\sigma_j)$ *satisfy (1.6), where* $\mu_j\neq\infty$. *Assume that for some* $\varkappa\times\varkappa$ *diagonal matrices* τ_d, ϑ_d *with diagonal elements* $\tau_1>0,\ldots,\tau_\varkappa>0$, $\vartheta_1,\ldots,\vartheta_\varkappa$, *respectively, we have*

$$\pi(\omega)\equiv\tau_d\mu_d^{-1}+2\mathrm{Re}(\tau_d+i\omega\vartheta_d)\chi(i\omega)>0,\ (-\infty<\omega<+\infty) \tag{4.20}$$

$$\lim_{\omega\to\infty}\omega^2\pi(\omega)>0, \tag{4.21}$$

where μ_d *is a* $\varkappa\times\varkappa$ *diagonal matrix with diagonal elements* $\mu_1,\ldots,\mu_\varkappa$. *Then, a) the linear part of system (1.2) is absolutely stable for the indicated class of nonlinearities; b) a Lyaupunov function can be constructed for system (1.2), in the form*

$$V=x^*Hx+\sum_{j=1}^{\varkappa}\vartheta_j\int_0^{\sigma_j}\varphi_j(\sigma_j)\,d\sigma_j. \tag{4.22}$$

Part a) was proved by Popov /12/ for the case in which (1.6) is replaced by $\varepsilon_j\sigma_j^2\leqslant\sigma_j\varphi_j(\sigma_j)\leqslant(\mu_j-\varepsilon_j)\sigma_j^2$.

For $\varkappa=1$ one may take $\tau_d=1$ without loss of generality, and then (4.20) coincides with (4.1). Other similar formulations of Popov's condition are known in which, under certain additional assumptions, the signs $>$ are replaced by $>$ (and, naturally, condition (4.21) is eliminated and the elements τ_j of the matrix τ_d may be zero) and, consequently, the parameter ϑ in condition (4.1) may take the value $\pm\infty$, etc.

Popov has also considered the simple critical case. The formulation of Theorem (4.1) does not change, except that (0.4) is replaced by (0.5) and the necessary condition for absolute stability

$$\Gamma^*=\operatorname*{Res}_{\lambda=0}\chi(\lambda)>0,$$

due to Lur'e, is added. In /25/, Popov's criterion is extended to the general critical case. Let us formulate the result of /25/ for the case of two purely imaginary roots.

Theorem 4.4. *Consider the system (1.2) with* $\varkappa = 1$ *and assume that equation (1.3) has two purely imaginary roots* $\pm i\omega_0 \neq 0$ *and that its remaining roots lie in the left half plane. Represent the function* $\chi(\lambda)$ *(see (1.5)) in the form* $\chi(\lambda) = \chi_1(\lambda) + \frac{\alpha\lambda+\beta}{\lambda^2+\omega_0^2}$ *where* $\chi_1(\lambda)$ *is holomorphic on the imaginary axis. For absolute stability of system (1.2) in the class of continuous functions* $\varphi(\sigma)$ *satisfying (0.5), it is sufficient that* $\alpha > 0$ *and*

$$\pi(\omega) \equiv \mu_0^{-1} + \frac{\beta}{\omega_0^2} + \mathrm{Re}\left[\left(1 + i\,\frac{\beta\omega}{\alpha\omega_0^2}\right)\chi_1(i\omega)\right] > 0 \tag{4.23}$$

for all $\omega > 0$, *if* $\lim\limits_{\omega\to\infty} \pi(\omega) = 0$; *in addition,*

$$\lim_{\omega\to\infty} \omega^2\pi(\omega) > 0.$$

We note that $\pi(\omega)$ can be represented in the form

$$\pi(\omega) = \mu_0^{-1} + \mathrm{Re}\left[\left(1 + i\,\frac{\beta\omega}{\alpha\omega_0^2}\right)\chi(i\omega)\right].$$

The inequality $\alpha > 0$ is a necessary condition for absolute stability.

§ 5. Solution of Lur'e's Problem. S-procedure

Let us return to the existence problem of a Lyapunov function of class **L** (i.e., of the form (2.11)) for system (1.2), for the principal case and $\varkappa = 1$, with an eye to the complete solution, i.e., we require not only sufficient but also necessary conditions for the existence of such a function.

Examining the proof of Theorem 3.2 (§ 3), one easily sees that when class $\mathbf{L}_0$ is considered instead of class **L** (see (2.2)) and $\mu_0 = \infty$ Lur'e's method gives not only sufficient but also necessary conditions. In fact, it is easily shown that condition (2.4) is not only sufficient but also necessary, and the proof of Theorem (3.2) implies that, without loss of generality, the matrix G may be assumed of the form $G = uu^* + \varepsilon I$.

In the general case a number of difficulties arise. For simplicity we confine ourselves to the case $\mu_0 \neq \infty$. The derivative of the function (2.11) has the form

$$-\dot{V} = x^*Gx + 2x^*g_0\varphi + \gamma_0\varphi^2, \tag{5.1}$$

where $\varphi = \varphi(\sigma)$, $g_0 = -\left(Hq + \frac{1}{2}\vartheta P^*r\right)$, $\gamma_0 = -\vartheta r^*q$.

It is natural to consider the following problems. It is required to find existence conditions for a matrix $H = H^*$ such that (I) $\dot{V} < 0$ for $x \neq 0$, where

$\varphi=\varphi\,(\sigma)$ is a fixed function satisfying (0.4) and $\sigma=r^*x$, (II) $\dot{V}<0$ for $x\neq 0$ for any function $\varphi=\varphi\,(\sigma)$ satisfying (0.4), (III) $\dot{V}$ is a negative definite form in the independent variables x, φ, where φ and σ are related by (0.4) or by

$$(\sigma-\varphi\mu_0^{-1})\,\varphi>0. \tag{5.2}$$

It is obvious that problems (II) and (III) coincide. If we assume that in problem (I) $\operatorname{Inf}\varphi\,(\sigma)/\sigma=0$ and $\sup\varphi\,(\sigma)/\sigma=\mu_0$ (where the Inf and sup are attained for $\mu_0\neq\infty$), then, as is easily checked, problem (I) is also equivalent to (II) and (III). One may thus consider problem (III), which is, however, fairly complicated.

To simplify the problem, we consider a special method (the S-procedure), first applied by Lure's /1/ and Rozenvasser /4/, which has also been found useful in a more general situation. The S-procedure for the general case is as follows. Let us assume that $\Phi\,(y)$ is a function of the vector argument y depending on certain parameters, conditions relating which must be found to ensure $\Phi\,(y)>0$ in the domain determined by the relations

$$y\neq 0,\quad \Omega_j(y)>0,\quad (j=1,\ldots,k), \tag{5.3}$$

where $\Omega_j\,(y)$ are certain functions depending on the same parameters. Let us set up the function

$$S\,(y)=\Phi\,(y)-\sum_{j=1}^{k}\tau_j\Omega_j\,(y)$$

(where $\tau_j>0$ are certain additional parameters) and require that, for values of the parameters $\tau_j>0$ and all $y\neq 0$, $S\,(y)>0$ be satisfied. Then, obviously, $\Phi\,(y)>0$ in the domain (5.3). The procedure thus reduces the problem of conditions that the function be positive-definite in the domain (5.3) to the analogous problem for all $y\neq 0$, which is usually simpler. In this case, however, simple examples show that the required domain in the parameter space may be larger than the domain obtained by the S-procedure. Nevertheless, it will follow from the sequel that these domains coincide in the case under consideration.* In our case $y=\|x,\varphi\|$, $k=1$, $\Omega_1=\varphi\,(r^*x-\varphi\mu_0^{-1})$ and, without loss of generality, $\tau_1=1$. Application of the S-procedure leads to the question of conditions for positive-definiteness of the expression in square brackets in (2.12). A solution of this problem is provided by Theorem 3.1 and leads to Popov's condition (4.1).

From part 2) of Theorem 3.3 it follows that if in addition the vectors

$$q,\ Pq,\ \ldots,\ P^{\nu-1}q \tag{5.4}$$

are linearly independent, a complete solution of the problem is also given by Lur'e's method, which leads to the resolvents (2.16).

If the S-procedure is not applied, the problem of positive-definiteness of the function (5.1) leads, putting $\varphi=\mu\sigma$, to the problem of existence conditions for a matrix $H = H^*$ such that

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* It would be important to find a general characterization of the classes of functions for which these domains coincide, i.e., of the classes for which the S-procedure does not "worsen" the result. For $k=1$ and quadratic forms the coincidence of these region follows from /9/.

$$x^*[G + (g_0 r^* + r g_0^*)\mu + \gamma_0 r r^* \mu^2] x > 0$$

for all $x \neq 0$ and $0 \leqslant \mu \leqslant \mu_0$. A solution of this problem is provided by Theorem 3.5, which yields a necessary and sufficient condition coinciding with (4.1).

Let us formulate the result /9/.

Theorem 5.1. *Assume that the vectors (5.4) are linearly independent. The following assertions are equivalent:*

a) There exists a Lyapunov function of class L *for system (1.2) which ensures absolute stability, such that* $\dot V < 0$ *for* $x \neq 0$.

b) Popov's frequency condition (4.1) is satisfied for some value of the parameter ϑ.

c) Lur'e's resolvents (2.16) have a real solution ϑ *for some value of the parameter* u.

As indicated, an analogous statement holds in the simple /3, 6/ and general critical /9, 25/ cases. An analogous statement is true also when the sign $>$ in (4.1) is replaced by $\geqslant$ and the values $\vartheta = \pm\infty$ are permitted, and for the Lyapunov function $\dot V \leqslant 0$.

§ 6. Absolute stability with given damping index

We require conditions on the linear part of system (1.2) which ensure that

$$|x(t)| \leqslant \beta e^{-\alpha t} |x(0)|, \tag{6.1}$$

where $\alpha > 0$ is given; the constants α and β depend only on the parameters of the linear part of the system and the numbers μ_j in formula (1.6). Thus the numbers α and β must not depend on the nonlinearities $\varphi_j(\sigma_j)$ or on $x(0)$. When (6.1) is satisfied, we shall say that system (1.2) is exponentially absolutely stable with damping index α.

Consider the case $\varkappa = 1$ and suppose that $\mu_0 \neq \infty$ and that the rank of each of the matrices

$$\| q, Pq, \ldots, P^{\nu-1} q \|, \quad \| r, P^* r, \ldots, P^{*\,\nu-1} r \| \tag{6.2}$$

is ν.* Since we may take $\varphi(\sigma) \equiv 0$ in (1.2), the roots of equation (1.3) must be assumed to lie in the half-plane $\operatorname{Re}\lambda \leqslant -\alpha$, and that the simple elementary divisors of the matrix P are the roots on the line $\operatorname{Re}\lambda = -\alpha$. Instead let us assume that the roots of equation (1.3) lie in the half-plane $\operatorname{Re}\lambda < -\alpha$.

Theorem 6.1. *Under the above assumptions, it is sufficient for exponential absolute stability with a damping index that for some* ϑ *and all* $\omega \geqslant 0$

$$\pi_\alpha(\omega) \equiv \mu_0^{-1} + \operatorname{Re}[(1 + i\omega\vartheta)\chi(-\alpha + i\omega)] \geqslant 0. \tag{6.3}$$

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* It is easily shown that this condition is equivalent to nondegeneracy of the function $\chi(\lambda)$, which means that $\chi(\lambda)$ cannot be expressed as the quotient of two polynomials where the denominator is of degree less than ν.

A similar statement, not assuming the conditions regarding the matrices (6.2), is proved in /32/; the sign $>$ in (6.3) is replaced by $>$ and the condition $+\infty > \lim_{\omega \to \infty} \omega^2 \pi_\alpha(\omega) > 0$ is added. In this case α may be replaced by $\alpha + \varepsilon$ with sufficiently small $\varepsilon > 0$ in (6.1). Theorem 6.1 is proved in exactly the same way, using Theorem 3.3 instead of Theorem 3.1

Inequality (6.3) has the same geometrical interpretation as Popov's criterion (see § 4).

There is an analogous statement for $\varkappa > 1$. Repeating the argument of /32/, we see that Theorem 6.1 remains valid also for $\varkappa > 1$ if condition (6.3) is replaced by

$$\tau_d u_d^{-1} + \operatorname{Re}\left[(\tau_d + i\omega\vartheta_d)\chi(-\alpha + i\omega)\right] > 0, \quad (-\infty < \omega < +\infty) \tag{6.4}$$

where τ_d and ϑ_d are diagonal matrices with diagonal elements $\tau_1 > 0, \ldots, \tau_\varkappa > 0, \vartheta_1, \ldots, \vartheta_\varkappa$ respectively, and all the numbers ϑ_j have the same signs (i.e., $\vartheta_j \vartheta_h > 0$).

§ 7. Systems with discontinuous nonlinearities. Absolute stability of the stationary set

Consider system (1.2) with $\varkappa = 1$ for simplicity, where $\varphi(\sigma)$ is a piecewise continuous function, such that for each discontinuity point σ_0 there are also finite limits

$$\varphi_-(\sigma_0) = \lim_{\substack{\sigma \to \sigma_0 \\ \sigma < \sigma_0}} \varphi(\sigma), \quad \varphi_+(\sigma_0) = \lim_{\substack{\sigma \to \sigma_0 \\ \sigma > \sigma_0}} \varphi(\sigma). \tag{7.1}$$

We retain the notation $\varphi_+(\sigma_0)$ and $\varphi_-(\sigma_0)$ also for points of continuity; then $\varphi_+(\sigma_0) = \varphi_-(\sigma_0) = \varphi(\sigma_0)$. Taking into account the possibility of sliding motion, we arrive at the following definition of the solution: a solution of the system (1.2) is any function $x(t)$, such that for some function $\psi(t)$*

$$\begin{gathered} \frac{dx}{dt} = Px + q\psi(t), \\ \varphi_-[\sigma(t)] \leqslant \psi(t) \leqslant \varphi_+[\sigma(t)], \quad \sigma(t) = r^* x. \end{gathered} \tag{7.2}$$

We shall call the function $\psi(t)$ the complementary function of $\varphi[\sigma(t)]$. It is obvious that for a continuous function $\varphi(\sigma)$, this definition agrees with the usual one. If $\varphi(\sigma)$ satisfies condition (0.4), then

$$0 \leqslant \sigma(t)\psi(t) \leqslant \mu_0 \sigma(t)^2 \quad (\mu_0 < +\infty). \tag{7.3}$$

We note that all the results of the present section remain valid for any other definition of the solution, when, for example, equations (1.2) are supplemented by the equations for the sliding motion /49/, provided (7.3) is satisfied for the complementary function.

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* The function $x(t)$ must be absolutely continuous, and $\psi(t)$ locally Lebesgue-integrable. Relations (7.2) must hold almost everywhere. This definition of the solution is in the sense of A.F. Filippov /48/.

Examining the proof of Theorem (4.1), which is based on the results of § 3, we see that it carries over with slight changes, if $\varphi[\sigma(t)]$ is replaced everywhere by $\psi(t)$ (see /24, 25/).

Theorem 7.1. *The assertion of Theorem 4.1 remains valid for the above class of discontinuous functions* $\varphi(\sigma)$.

A similar theorem was established by Tsypkin /15/, by a different method.

The argument of § 4 leading to condition (4.4) make essential use of the continuity of $\varphi(\sigma)$. This is not accidental: in the case of a discontinuous function, condition (4.4) does not guarantee absolute stability. In fact, if $\sigma = 0$ is a discontinuity point of the function $\varphi(\sigma)$ and $\chi(0) = 0$, the system (1.2) has a "stationary interval"

$$x = -P^{-1}q \cdot \psi, \quad \psi \in [\varphi_-(0), \varphi_+(0)] \tag{7.4}$$

(for the stationary solution (7.4) the complementary function $\psi(t) \equiv \psi$). Thus the solution $x \equiv 0$ is not absolutely stable, although (4.4) may be satisfied. Nevertheless, in this case one may define absolute stability of the stationary set (7.4).

Various types of stability of the stationary set are possible: an arbitrary solution may tend to some point of the stationary set, or may converge to the stationary set in the sense of approaching it without limit, while not tending to any of its points. Let us give the rigorous definitions /30/.

Consider the system

$$\frac{dx}{dt} = f(x), \tag{7.5}$$

where x and $f(x)$ are ν-dimensional real vectors, and $f(x)$ is measurable and bounded in any finite part of the space $\{x\}$. A solution will be understood in the sense of Filippov /48/ and the stationary set (the set of equilibrium positions) will be assumed bounded, connected and closed.

A stationary set Λ is called stable in the large, if it is Lyapunov-stable "in the small" and

$$\rho[x(t), \Lambda] = \min_{z \in \Lambda} |x(t) - z| \to 0, \text{ for } t \to \infty.$$

If Λ is stable in the large and, in addition, when $t \to \infty$ each motion $x(t)$ tends to a definite equilibrium position $z \in \Lambda$, we shall say that the stationary set Λ is pointwise stable in the large.

Gelig has established the following lemma of the Lyapunov type:

Lemma 1 /30/. *Suppose that for any* $c \in \Lambda$ *and any vector* x *there exists a continuous function* $V(x, c)$ *satisfying the following conditions:*

(I) $V(x, c) > 0$ *for* $x \notin \Lambda$, $\lim\limits_{x \to c} V(x, c) = 0$ *and* $\lim\limits_{|x| \to \infty} V(x, c) = \infty$;

(II) there exists an ε*-neighborhood of the set* Λ, *in which for any* $c \in \Lambda$ *the function* $V(x, c)$ *does not increase along any trajectory of system (7.5);*

(III) there exists $c_0 \in \Lambda$, *such that* $V(x, c_0)$ *does not increase along any trajectory, and on any surface* $V(x, c_0) = \text{const}$ *there are no other positive semi-trajectories of system (7.5), apart from the equilibrium positions.*

Then the stationary set Λ *of the system (7.5) is stable in the large.*

If in addition to conditions (I)—(III) the function $V(x, c)$ *has the property (IV)* $V(x, c) = 0$ *for any* $c \in \Lambda$ *only for* $x = c$, *then the stationary set is pointwise stable in the large.*

Application of this lemma to system (1.2) using the matrix inequality theorems of § 3 yields the following result /30/.

Theorem 7.2. *Suppose* $\varkappa = 1$, *the principle case obtains and the following conditions are satisfied:*

(I) The function $\varphi(\sigma)$ *has a discontinuity of the first kind at* $\sigma = 0$, *at continuity points it satisfies inequality (2.1), and there exists* $\varepsilon > 0$ *such that* $\varphi(\sigma) > \varphi_+(0)$ *for* $0 < \sigma < \varepsilon$, *and* $\varphi(\sigma) \leqslant \varphi_-(0)$ *for* $-\varepsilon < \sigma < 0$;

(II) $\chi(0) = 0$, $\chi(i\omega) \neq 0$ *for* $\omega \neq 0$;

(III) the rank of each of the matrices (6.2) is ν;

(IV) there exists $\vartheta > 0$ *such that for all* $\omega > 0$

$$\operatorname{Re}[(1 + i\omega\vartheta)\chi(i\omega)] > 0 \tag{7.6}$$

and if $\vartheta > 0$ *the number* $-\vartheta^{-1}$ *is not an eigenvalue of the matrix* P.

Then the stationary interval (7.4) is pointwise stable in the large.

In /25/ the general critical case is considered, when $x = 0$ is the only stationary state.

The research of Gelig and Komarnitskaya, soon to be published, deals with the general critical case in the presence of a stationary interval.

The case $\varkappa > 1$ is considered analogously. Theorem 4.3 is also valid without alteration for the case of discontinuous nonlinearities.

§ 8. Systems with hysteresis nonlinearities

Consider the system

$$\frac{dx}{dt} = Px + q\varphi[\sigma, \varphi_0]_t, \quad \sigma = r^*x \tag{8.1}$$

in the principal case and, first, for $\varkappa = 1$. Here $\varphi[\sigma, \varphi_0]_t$ is a hysteresis function. The value of $\varphi[\sigma, \varphi_0]_t$ is the output of the nonlinear unit at the instant t in response to the input $\sigma(\tau), 0 \leqslant \tau \leqslant t$; it depends, in addition, on the state φ_0 of the nonlinear unit at the initial instant $t = 0$. More formally, let us assume: 1) with each σ_0, $-\infty < \sigma_0 < +\infty$ we associate a set $E[\sigma_0]$ of "initial values of the hysteresis function"; 2) with each $t > 0$, each function $\sigma(\tau)$ continuous in $[0, t]$ and any $\varphi_0 \in E[\sigma(0)]$ we associate a number $\varphi[\sigma, \varphi_0]_t$; 3) $\varphi[\sigma, \varphi_0]_0 = \varphi_0$. We shall then say that the hysteresis function $\varphi[\sigma, \varphi_0]_t$ is given. If the number $\varphi[\sigma, \varphi_0]_t$ is a continuous function of the two numerical arguments t and $\sigma(t)$, we shall say that the hysteresis function is continuous. The simplest example of a continuous hysteresis function is provided by backlash (Figure 2). Here $\sigma(t)$ is the abscissa of the point A, $E[\sigma_0] = [\sigma_0, \sigma_0 + \delta]$, and $\varphi[\sigma, \varphi_0]_t$ is the abscissa of the point B. Figures 3 and 4 give with the usual interpretation graphs of some continuous hysteresis function. More general definitions are possible, according to which $\varphi[\sigma, \varphi_0]_t$ is in general a discontinuous hysteresis function. In this case the function $\varphi[\sigma, \varphi_0]_t$ in (8.1) is a complementary function, in analogy with the case of ordinary discontinuous functions (see § 7).

The vector hysteresis function is defined in a similar manner: in this case $\varphi\,[\sigma, \varphi_0]_t = \|\varphi_j\,(\sigma_j, \varphi_{0j})\|_t$ is a $\varkappa \times 1$ vector. We shall assume that for each scalar function $\varphi_j[\sigma_j, \varphi_{0j}]_t$ we have

$$0 \leqslant \sigma_j(t)\varphi_j\,[\sigma_j, \varphi_{0j}]_t \leqslant \mu_j\sigma_j(t)^2 \quad j = 1, \ldots, \varkappa \tag{8.2}$$

for any $t > 0$. This condition means, in descriptive terms, that the graph of the hysteresis function of $\varphi_j\,[\sigma_j, \varphi_{0j}]_t$, while generally varying in time, always remains in a fixed region of the $\{\sigma, \varphi\}$ plane between the lines $\sigma = 0$ and $\varphi = \mu_j\sigma$. The backlash (see Figure 2) does not satisfy condition (8.2) with $\mu_j = \mu_0$, whereas the functions in Figures 3 and 4 do.

Theorem 8.1. *Suppose that for some diagonal $\varkappa \times \varkappa$ matrix τ_d with positive diagonal elements and for all $\omega > 0$**

$$\tau_d\mu_d^{-1} + \mathrm{Re}\,[\tau_d\chi\,(i\omega)] > 0, \quad (-\infty < \omega < +\infty) \tag{8.3}$$

where μ_d is a $\varkappa \times \varkappa$ diagonal matrix with diagonal elements $\mu_1, \ldots, \mu_\varkappa$. Then the linear part of the system (8.1) is exponentially absolutely stable for the class of nonlinearities satisfying relations (8.2), i.e., there exist constants $\varepsilon > 0$, $\gamma > 0$ depending only on $P, q, r, \mu_1, \ldots, \mu_\varkappa$, such that for any solution of system (8.1) and any $t > t_0$

$$|x(t)| \leqslant \gamma e^{-\varepsilon(t-t_0)}|x(t_0)|. \tag{8.4}$$

Proof. For the function $V(x) = x^*Hx$, where $H = H^*$ is the required matrix, we have $\dot{V} = 2x^*H\,(Px + q\varphi)$.

Here $\varphi = \varphi\,[\sigma, \varphi_0]_t$. Applying the S-procedure, we transform $\dot{V}$ by

$$-\dot{V} = [x^*Gx + 2x\,g\varphi + \varphi^*\tau_d\mu_d^{-1}\varphi] + \Omega, \tag{8.5}$$

where

$$\Omega = \sum_{j=1}^{\varkappa}\tau_j\varphi_j(\sigma_j - \varphi_j\mu_j^{-1}), \quad g = -(Hq + r\tau_d), \quad G = -(P^*H + HP).$$

Since $\tau_j > 0$, we have from (8.2) that $\Omega > 0$. By Theorems 3.4 and 3.1 it follows from inequality (8.3) that there exists a matrix $H = H^*$ such that the expression in square brackets in (8.5) is a positive-definite form in x and φ. In this case $G > 0$ implies $H > 0$. From (8.5) we have, for some $\varepsilon > 0$,

$$-\dot{V} \leqslant 2\varepsilon V \tag{8.6}$$

Integration of (8.6) yields (8.4).

The above proof, as well as Theorem 8.1 may be refined at a number of points (under certain additional conditions, the sign $>$ in (8.3) may be replaced by $\geqslant$, τ_j and μ_j may take the values 0 and ∞ respectively). For $\varkappa = 1$ condition (8.3) obviously takes the form $\mu_1^{-1} + \mathrm{Re}\,\chi\,(i\omega) > 0$. For $\varkappa = 2$, when $|\chi_{12}| \equiv \mathrm{const}\,|\chi_{21}|$ the matrix condition (8.3) is transformed, as is easily verified, into

$$\left.\begin{aligned} &\mu_1^{-1} + \mathrm{Re}\,\chi_{11}\,(i\omega) > 0, \quad \mu_2^{-1} + \mathrm{Re}\,\chi_{22}(i\omega) > 0 \\ &2\,[\mu_1^{-1} + \mathrm{Re}\,\chi_{11}(i\omega)]\,[\mu_2^{-1} + \mathrm{Re}\,\chi_{22}(i\omega)] > \\ &> |\chi_{12}(i\omega)|\cdot|\chi_{21}(i\omega)| + \mathrm{Re}\,[\chi_{12}(i\omega)\cdot\chi_{21}(i\omega)]. \end{aligned}\right\} \tag{8.7}$$

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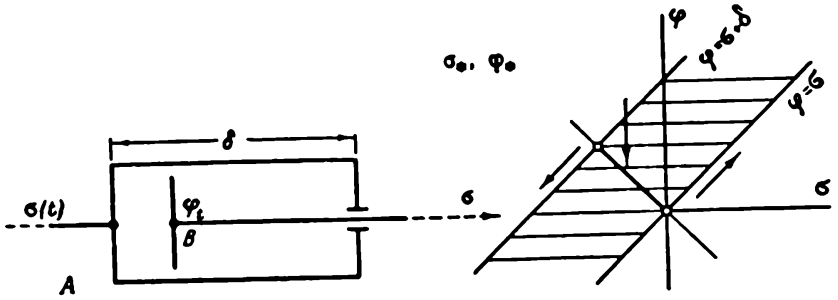
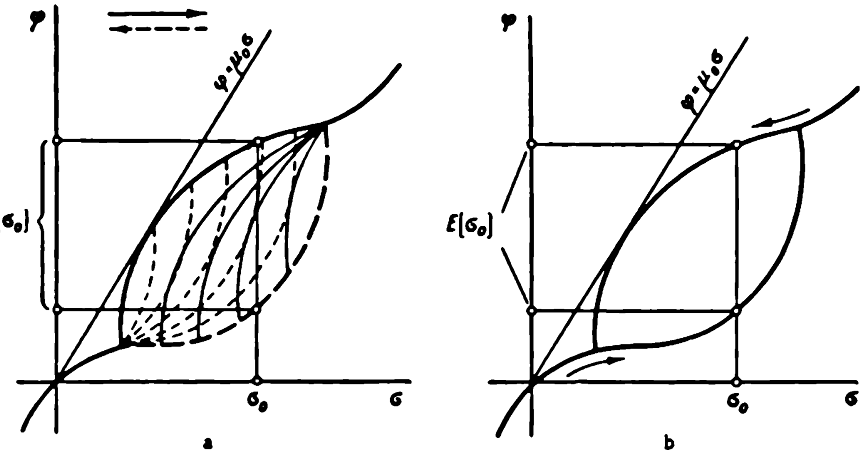
* Here, we recall, $\mathrm{Re}Z = \frac{1}{2}(Z + Z^*)$. The left-hand side of inequality (8.3) is a $\varkappa \times \varkappa$ matrix.


σ₀, φ₀
δ
σ(t)
φ_t
B
A
σ
φ
φ=σ-δ
φ=σ
σ

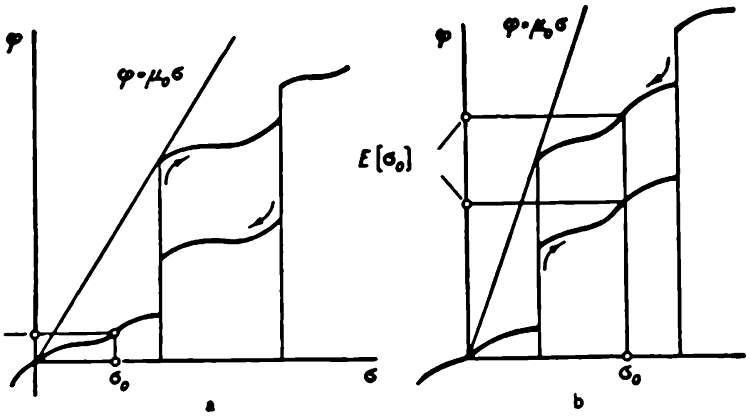
2


φ
φ=μ₀σ
E[σ₀]
σ₀
σ
φ
φ=μ₀σ
E[σ₀]
σ₀
σ

a

b

3


φ
φ=μ₀σ
E[σ₀]
σ₀
σ
φ
φ=μ₀σ
E[σ₀]
σ₀

a

b

4

For $\varkappa = 2$, when $|\chi_{12}| \not\equiv \mathrm{const}\,|\chi_{21}|$, we must have $\max_{\omega} \eta_1(\omega) \leqslant \min_{\omega} \eta_2(\omega)$ in addition to (8.7), where $\eta_1 \leqslant \eta_2$ are roots of the equation

$$\eta\,|\chi_{12}|^2 + \eta^{-1}\,|\chi_{21}|^2 - 4(\mu_1^{-1} + \operatorname{Re}\chi_{11})(\mu_2^{-1} + \operatorname{Re}\chi_{22}) + 2\operatorname{Re}(\chi_{12}\chi_{21}) = 0.$$

We note that Theorem 8.1 is also applicable when some of the "nonlinear units" are "linear nonstationary", i.e., when $\varphi_j[\sigma_j, \varphi_{0j}]_t = \alpha_j(t)\sigma_j$ for certain j, where $\alpha_j(t)$ are any functions satisfying (by virtue of (8.2)) the condition $0 \leqslant \alpha_j(t) \leqslant \mu_j$.

For $\varkappa = 1$, Theorem 8.1 was proved independently by Popov /26/ (in a similar formulation, but without the estimate (8.4)) and by one of the authors /27/. The above proof resembles that of /27/.

Formulation of additional results uses certain properties of hysteresis functions.

Definitions: *1) A hysteresis function* $\varphi[\sigma, \varphi_0]_t$ is called finitely-bounded, if for any $\gamma > 0$ there exists $\gamma_1 > 0$ such that the $|\sigma(t)| \leqslant \gamma$ implies $|\varphi[\sigma, \varphi_0]_t| < \gamma_1$ for any t and $\sigma(t)$.*

2) A hysteresis function is said to satisfy the positiveness condition if for any function $\sigma(t)$ and any $\varphi_0 \in E[\sigma_0]$ there exists a constant $\delta > 0$, such that

$$\int_0^t \varphi[\sigma, \varphi_0]_\tau d\sigma(\tau) > -\delta \text{ for } t > 0,$$

and the strong positiveness condition if there exists a nondecreasing function $\delta(\sigma)$, defined in $[0, \infty)$, such that $\delta(0) = 0$ and for any $\sigma(t)$ and $\varphi_0 \in E[\sigma_0]$ we have

$$\int_0^t \varphi[\sigma, \varphi_0]_\tau d\sigma(\tau) > -\delta[\sigma(0)].$$

3) A hysteresis function is called continuous in the limit, if the relations $\sigma(t) \to \sigma_*$, $\varphi[\sigma, \varphi_0]_t \to \varphi_*$ for $t \to \infty$ imply that $\varphi_* \in E[\sigma_*]$ and $\varphi[\sigma_*, \varphi_*] \equiv \varphi_*$.

We illustrate these definitions with the following examples. $\varphi[\sigma, \varphi_0]_t = \alpha(t)\sigma^3$, for $|\alpha(t)| < \mathrm{const}$, is a finitely bounded function satisfying the positiveness condition if $\alpha(t) > 0$ and $d\alpha(t)/dt \leqslant 0$. It is not continuous in the limit if $\alpha(t) \not\equiv \mathrm{const}$.

The backlash is continuous in the limit. The functions in Figures 3a and 4a satisfy the positiveness condition, while those in Figures 3b and 4b do not. In the latter case, however, the functions $\mu_0\sigma - \varphi[\sigma, \varphi_0]_t$ do satisfy the condition.

Condition 2) means, in descriptive, imprecise terms, that the area increment along any hysteresis loop is nonnegative for the function $\varphi[\sigma, \varphi_0]_t$.

The backlash, a hysteresis function important in applications, does not satisfy condition (8.2). When condition (8.2) is not satisfied, the system (8.1) may have a nonunique equilibrium state. Stationary solutions of the system (8.1) exist for $\det P \neq 0$ and $\det \chi(0) \neq 0$, as is easily verified in the following manner.

---

* We are considering a scalar function.

Let $\sigma = \sigma_*$ and $\varphi = \varphi_*$ be any stationary solutions of the equations

$$\sigma_* + \chi(0)\varphi_* = 0, \quad \varphi[\sigma_*, \varphi_0]_t \equiv \varphi_*. \tag{8.8}$$

(If $\varkappa = 1$ and $\varphi[\sigma, \varphi_0]_t$ is the backlash, then (σ_*, φ_*) are the intersection points of the backlash graph with the characteristic line $\sigma + \chi(0)\varphi = 0$ — see Figure 3). The set of stationary solutions of (8.1) is the set of all $c = -P^{-1} q\varphi_*$, where φ_* runs through all the values obtained as above.

Theorem 8.2. *Suppose that the conditions of Theorem 8.1 are satisfied, with condition* (8.2) replaced by*

$$0 < \frac{d\sigma_j}{dt} \cdot \frac{d\varphi_j[\sigma_j, \varphi_{0j}]_t}{dt} \leqslant \mu_j\left(\frac{d\sigma_j}{dt}\right)^2, \quad j = 1, \ldots, \varkappa. \tag{8.9}$$

Then as $t \to \infty$ the solution $x(t)$ and the value of the vector hysteresis function $\varphi[\sigma, \varphi_0]$ tend to certain limits

$$x(t) \to x_\infty, \quad \varphi[\sigma, \varphi_0]_t \to \varphi_\infty. \tag{8.10}$$

This convergence is exponential, and moreover there exist constants γ_j, $\varepsilon > 0$ depending on the linear part of the system (but not on $\varphi[\sigma, \varphi_0]_t$ or on the initial values $x(0), \varphi_0$), such that

$$\left.\begin{aligned} |x(t) - x_\infty| &\leqslant [\gamma_1|x(0)| + \gamma_2|\varphi_0|]\,e^{-\varepsilon t}, \\ |\varphi[\sigma, \varphi_0]_t - \varphi_\infty| &\leqslant [\gamma_3|x(0)| + \gamma_4|\varphi_0|]\,e^{-\varepsilon t}. \end{aligned}\right\} \tag{8.11}$$

If $\varphi_j[\sigma_j, \varphi_{0j}]_t$ are continuous in the limit, then the limits are stationary solutions of system (8.1).

Theorem 8.3. Under the assumptions of Theorems 8.1 and 8.2, if all the roots of the equation $\det(P - \lambda I) = 0$ lie in the half-plane $\mathrm{Re}\,\lambda < -\alpha < 0$, the rank of each of the matrices (6.2) is τ_d and for some diagonal matrix with positive diagonal elements we have

$$\tau_d\mu_d^{-1} + \mathrm{Re}[\tau_d\chi(-\alpha + i\omega)] > 0 \text{ for } \omega > 0, \tag{8.12}$$

then in (8.4) and (8.11) we may take $\varepsilon = \alpha$.

The proofs of Theorems 8.2 and 8.3 coincide word for word with those of Theorems 3 and 4 of /28/, where the case $\varkappa = 1$ is considered. To prove Theorem 8.2, one must differentiate system (8.1) with respect to time and apply Theorem 8.1. From the estimate (8.4) one easily derives (8.11).

We note that when $\varphi[\sigma, \varphi_0]_t$ is the backlash, condition (8.9) is satisfied with $\mu_j = 1$.

We formulate the following theorem /27, 28/, for simplicity, for $\varkappa = 1$.

Theorem 8.4. *Suppose $\varkappa = 1$,*

$$0 < \sigma(t)\varphi[\sigma, \varphi_0]_t < \mu_0\sigma(t)^2, \quad (\mu_0 < \infty)$$

---

* The case when instead of (8.9) we have $\mu_j^{(1)}\sigma_j^2 < \varphi_j\sigma_j < \mu_j^{(2)}\sigma_j^2$ may be reduced to the case under consideration by the substitution $\varphi_j^{(1)} = \varphi_j - \mu_j^{(1)}\sigma_j$.

and $\mu_0 = \infty$; *then* $\varphi\,[\sigma, \varphi_0]_t$ *is a finitely bounded function.*[*]

If $\varphi\,[\sigma, \varphi_0]_t$ *satisfies the positiveness condition and (4.1) holds for some* $\vartheta > 0$ *or (for* $\mu_0 \neq \infty$*) if* $\mu_0\sigma - \varphi\,[\sigma, \varphi_0]_t$ *satisfies the positiveness condition and (4.1) holds for some* $\vartheta \leqslant 0$, *then*

$$x(t) \to 0 \text{ for } t \to \infty, \qquad 8.13)$$

where $x(t)$ *is any solution of system (8.1).*

If the positiveness condition is replaced by the strong positiveness condition in the above condition, system (8.1) is absolutely stable, and for $\mu_0 \neq \infty$ *it is uniformly absolutely stable.*[**]

This theorem shows that Popov's condition remains valid in the case of a hysteresis nonlinearity, if the parameter ϑ has a definite sign depending on the direction in which the hysteresis loop is circulated.

§ 9. Differentiable nonlinearities strengthening of Popov's frequency condition

The results of § 5 imply that Popov's criterion cannot be strengthened by using Lyapunov functions of class **L**. Consider (for $\varkappa = 1$) the class $\mathbf{L}_1$ of functions

$$V = x^*Hx + 2x^*h\varphi + \gamma\varphi^2 + \vartheta \int_0^\sigma \varphi d\sigma, \qquad (9.1)$$

where H is a $\nu \times \nu$ symmetric matrix, h is $\nu \times 1$ vector, γ and ϑ are numbers. We shall assume that $\varphi(\sigma)$ is differentiable and that in addition to (0.3) we have

$$-\alpha_1 < \varphi'(\sigma) < \alpha_2, \qquad (9.2)$$

where $\alpha_1 \neq \infty, \alpha_2 \neq \infty$ and without loss of generality $\alpha_1 > 0, \alpha_2 > \mu_0$.

The problem of existence conditions for a Lyapunov function of class $\mathbf{L}_1$ for system (1.2) can be reduced, using the S-procedure, to certain new matrix inequalities /9/, similar to those of § 3; their solution yields the following result /32, 33/.

Theorem 9.1. *Consider system (1.2) for* $\varkappa = 1$ *in the principal case. If there exist* $\tau_1 > 0$ *and* $\tau_2 > 0$ *such that*

$$+\infty > \tau_2 \varliminf_{\sigma \to \pm\infty} \frac{\vartheta}{\sigma^2}\left[\int_0^\sigma \varphi(\sigma)\,d\sigma - \frac{\sigma\varphi(\sigma)}{2}\right] > 0, \qquad (9.3)$$

and such that for all $\omega > 0$

$$\begin{aligned}\pi_1(\omega) \equiv \tau_1\,[\mu_0^{-1} + \operatorname{Re}\chi(i\omega)] + \vartheta \operatorname{Re}[i\omega\chi(i\omega)] + \\ + \tau_2\omega^2\,[1 + (\alpha_2 - \alpha_1)\operatorname{Re}\chi(i\omega) - \alpha_1\alpha_2\,|\chi(i\omega)|^2] > 0,\end{aligned} \qquad (9.4)$$

then system (1.2) is asymptotically stable in the large.

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[*] For $\mu_0 < \infty$ this condition is satisfied automatically. If $\rho_0 = \varliminf_{\omega \to \infty} \pi(\omega) > 0$, where $\pi(\omega)$ is defined by formula (4.1), the assumption of boundedness may be omitted.

[**] For the definition of absolute stability and uniform absolute stability see § 1.

An analogous statement is valid for the case when one of the numbers α_1, α_2, and μ_0 is ∞*. Then $\pi_1(\omega)$ has the form

$$\pi_1(\omega) \equiv \tau_1[\mu_0^{-1} + \mathrm{Re}\,\chi(i\omega)] + \vartheta\,\mathrm{Re}[i\omega\chi(i\omega)] + \varepsilon_0\tau_2\omega^2[\mathrm{Re}\,\chi(i\omega) + \alpha|\chi(i\omega)|^2],$$

where $\alpha = -\alpha_1$, $\varepsilon_0 = 1$, if $\alpha_2 = \infty$, $\alpha = \alpha_2$, $\varepsilon_0 = -1$, if $\alpha_1 = \infty$.

In this case the condition $+\infty > \lim\limits_{\omega\to\infty} \omega^2\pi_1(\omega) > 0$ must be satisfied in addition to (9.4).

Under certain additional assumptions, an analogous statement is valid, with the sign $>$ in (9.4) replaced by $\geqslant$ and the conditions $\tau_1 > 0$, $\tau_2 > 0$ by the conditions $\tau_1 \geqslant 0$, $\tau_2 > 0$ (see /33/).

Condition (9.3) restricts the behavior of the function $\varphi(\sigma)$ only at infinity; it is satisfied if, for example, there exist finite or infinite limits $\lim\limits_{\sigma\to\pm\infty} \varphi(\sigma)/\sigma$.

Popov's condition is obtained from (9.4) by putting $\tau_2 = 0$. The problem of existence of the parameters $\tau_1, \tau_2, \vartheta$ has a simple geometrical solution. We denote

$$\zeta(\omega) = \omega^2[1 + (\alpha_2 - \alpha_1)\,\mathrm{Re}\,\chi(i\omega) - \alpha_1\alpha_2|\chi(i\omega)|^2],$$

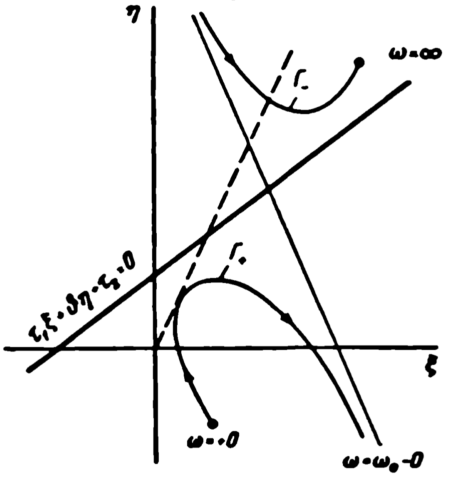
$$\xi(\omega) = \frac{1}{\zeta(\omega)}\left[\mu_0^{-1} + \mathrm{Re}\,\chi(i\omega)\right], \quad \eta(\omega) = \frac{1}{\zeta(\omega)}\mathrm{Re}\left[i\omega\chi(i\omega)\right] \tag{9.5}$$

and construct in the $\{\xi, \eta\}$ plane the curves**

$$\Gamma_+\{\xi = \xi(\omega),\ \eta = \eta(\omega),\ \zeta(\omega) > 0\},$$

$$\Gamma_-\{\xi = \xi(\omega),\ \eta = \eta(\omega),\ \zeta(\omega) < 0\}.$$

We assume for simplicity that there are no values $\omega = \omega_0$ for which simultaneously $\xi(\omega_0) = \eta(\omega_0) = \zeta(\omega_0) = 0$. Any straight line in the $\{\xi, \eta\}$ plane which intersects the half-line $\eta = 0$, $\xi \leqslant 0$ will be called an admissible line. The open half-plane bounded by an admissible line and containing the half-line $\eta = 0$, $\xi > 0$ will be called a positive half-plane; the complementary open half-plane will be called a negative half-plane. We shall say that an admissible line separates the curves Γ_+ and Γ_- if the whole curve Γ_+ lies in the positive, and the curve Γ_- in the negative half-plane. The frequency condition (9.4) is satisfied (i.e., the required values of the parameters τ_1, τ_2, and ϑ exist) if there exists an admissible line separating the curves Γ_+ and Γ_- (Figure 5, where the heavy line is admissible). Popov's frequency condition means, in terms of the curves Γ_+ and Γ_-



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* When $\mu_0 = \infty$, obviously also $\alpha_2 = \infty$.

** The values of ω satisfy the inequality $\zeta(\omega) > 0$, on the curve Γ_+ and the inequality $\zeta(\omega) < 0$ on the curve Γ_-.

that there exists an admissible line which separates the curves Γ_* and Γ and passes through the origin. Figure 5 illustrates a case in which a separating line can be drawn, but not through the origin of the coordinates. For such cases Popov's frequency condition is not satisfied, while absolute stability may be established by the criterion formulated above.

A condition analogous to (9.4) can be obtained for the case $\varkappa > 1$.

In /32/ it is shown that for $\alpha_1 = \infty$ or $\alpha_2 = \infty$, the above frequency condition includes all conditions obtainable by using Lyapunov functions of class $\mathbf{L}_1$. For finite α_1 and α_2 the problem remains open; it reduces to justification of the S-procedure for the given case.

§ 10. Systems with external action. Dissipative systems. Existence of periodic motions.

Consider the system

$$\frac{dx}{dt} = f(t,\ x), \tag{10.1}$$

where $f(t,\ x)$ is a continuous function in the $(\nu + 1)$-dimensional space $\{t,\ x\}$. The system (10.1) is called, as is well-known, dissipative, if:

(I) There exists a bounded closed set $\{x\}$ in the ν-dimensional space which is invariant for solutions of system (10.1), i.e., a set such that $x(t_0) \in \mathrm{F}$ implies $x(t) \in \mathrm{F}$ for $t > t_0$;

(II) for any solution $x(t)$ there exists t_0 such that $x(t_0) \in \mathrm{F}$.

It can be shown that every dissipative system has a solution $x^0(t) \in \mathrm{F}$ for $-\infty < t < +\infty$. The boundedness of the set F implies the boundedness of the solution $x^0(t)$ in $(-\infty,\ +\infty)$. Thus, dissipative systems are systems possessing, in some weak sense, limit motions, of which, as simple examples show, there may be several.

The following general proposition of the Lyapunov type, implicitly contained in the work of N.N. Luzin /50/ and B.P. Demidovich /51, 52/, is similar to certain results of Yoshizawa /39/.

Lemma 2. *Suppose there exists a differentiable function* $V(x)$ *with the properties:*

(I) $V(x) \to \infty$ *for* $|x| \to \infty$;

(II) outside some sphere $|x| > \xi$

$$\dot V \equiv \frac{\partial V}{\partial x} \cdot f(t,\ x) \leqslant -\alpha(x) < 0,$$

where $\alpha(x)$ *is a continuous function. Then the system (10.1) is dissipative, and for the set* F *one may take the set* F_η *of all* x *for which* $V(x) < \eta$*, provided that* $\eta > 0$ *is such that the set* F_η *contains the sphere* $|x| \leqslant \xi$.

Consider system (1.1) in the principle case and, for simplicity, with $\varkappa = 1$. Application of Lemma 2, the function (2.11) playing the role of $V(x)$ yields (using the theorems of § 3) the following result /31/.

Theorem 10.1. *Assume that (I) in system (1.1) the function* $f(t)$ *is bounded in* $(0,\ \infty)$; *(II) for some* ϑ *conditions (4.1) and (4.2), where* $0 < \mu_0 \leqslant +\infty$, *are satisfied; (III) the function* $\varphi(\sigma)$ *in (1.1) satisfies the relation*

$$\varliminf_{|\sigma|\to\infty} \frac{\varphi(\sigma)}{\sigma}\left[1 - \frac{1}{\mu_0} \cdot \frac{\varphi(\sigma)}{\sigma}\right] > 0, \tag{10.2}$$

and, if $\vartheta \neq 0$, $\lim\limits_{\omega\to\infty} \pi(\omega) = 0$, *then also*

$$\lim_{|\sigma|\to\infty} \frac{\varphi(\sigma)}{\sigma} = 0.$$

Then system (1.1) is dissipative.

Condition (10.2) is obviously considerably more general than condition (0.4).

Theorem 10.2. *Assume that in system (1.1) the function* $f(t)$ *is* T*-periodic and that the conditions of Theorem 10.1, with (10.2) replaced by (0.4), are satisfied. Then system (1.1) has a* T*-periodic solution.*

The proof is based on the application of Brouwer's principle to the set $\mathbf{F}_n$, which can be shown in this case to be homeomorphic to a sphere.

Theorem 10.3. *Assume that in (1.1) the function* $f(t)$ *is bounded in* $(0, \infty)$, *and* $\varphi(\sigma)$ *is differentiable and satisfies conditions (0.4) and (9.2). If for some* $\tau_1 > 0$, $\tau_2 > 0$, *and* ϑ *(9.4) is satisfied, then system (1.1) is dissipative.*

This is proved by applying Lemma 2 with the function (9.1) and using Theorem 4 /33/, which is similar to Theorem 3.1 and concerns a matrix inequality.

Analogous statements are valid for $\varkappa > 1$.

§ 11. Absolutely stable periodic and almost-periodic limit motions

We first consider stability in the large of some fixed solution $x^0(t)$ of system (1.1) with $\varkappa > 1$ and continuous functions $\varphi_j(\sigma_j)$. Putting $y = x(t) - x^0(t)$, where $x(t)$ is any other solution, we have from (1.1)

$$\frac{dy}{dt} = Py + q\varphi_1(\sigma_1, t), \quad \sigma_1 = r^* y, \tag{11.1}$$

where $\varphi_1(\sigma_1, t) = \varphi[\sigma^{(0)}(t) + \sigma_1] - \varphi[\sigma^{(0)}(t)], \sigma^{(0)}(t) = r^* x^0(t)$.

Let

$$0 < \frac{\varphi_j(\sigma_j) - \varphi_j[\sigma_j^{(0)}(t)]}{\sigma_j - \sigma_j^{(0)}(t)} < \mu_j, \quad j = 1, \ldots, \varkappa \tag{11.2}$$

for all σ_j, $-\infty < \sigma_j < +\infty$ and $t > t_*$, where $t_* = -\infty$ is possible. Applying Theorem 8.1 to system (11.1) we see that when the frequency condition (8.3) is satisfied, we have for all $t > t_0 > t_*$

$$|x(t) - x^0(t)| \leqslant \gamma e^{-\varepsilon(t-t_0)} |x(t_0) - x^0(t_0)|, \tag{11.3}$$

where $\gamma > 0$ and $\varepsilon > 0$ are constants determined only by the parameters of the linear part of the system and the numbers μ_j.

Condition (11.2) is automatically satisfied if for all $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$ we have*

$$0 \leqslant \frac{\varphi_j(\sigma_j^{(1)}) - \varphi_j(\sigma_j^{(2)})}{\sigma_j^{(1)} - \sigma_j^{(2)}} \leqslant \mu_j, \quad j = 1, \ldots, \varkappa. \tag{11.4}$$

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* In this case, obviously, we may take $t_* = -\infty$.

In the frequently encountered case when the graph of $\varphi_j(\sigma_j)$ consists of straight line segments with (without loss of generality) nonnegative slopes, the number μ_j is the maximum slope. If the function $\varphi_j(\sigma_j)$ is differentiable, condition (11.4) is equivalent to

$$0 \leqslant d\varphi_j(\sigma_j)/d\sigma_j \leqslant \mu_j, \quad j = 1, \ldots, \varkappa. \tag{11.5}$$

In accordance with the customary terminology, we shall speak of exponential convergence for system (1.1) if for any two solutions $x(t)$ and $\hat{x}(t)$ of system (1.1) and any $t > t_0$ the estimate (11.3) holds for fixed numbers $\gamma > 0$, $\varepsilon > 0$. The above considerations yield:

Theorem 11.1. *Suppose that (11.4) and the frequency condition (8.3) are satisfied, where* τ_d *is any diagonal* $\varkappa \times \varkappa$ *matrix with positive diagonal elements and* μ_d *is a diagonal* $\varkappa \times \varkappa$ *matrix with diagonal elements* $\mu_1, \ldots, \mu_\varkappa$. *Then system (1.1) has the exponential convergence property.**

For $\varkappa = 1$ a similar theorem is proved by Tsypkin and Naumov /16/. We note that under the conditions of Theorem 11.1 the solutions may not be bounded in $(0, \infty)$ (if, for example, the function $f(t)$ is not bounded in $(0, \infty)$).

Using Theorem 10.1 and additional considerations based on the estimate (11.3), the following result can be obtained (the proof is given in /31/ for $\varkappa = 1$):

Theorem 11.2. *Suppose that the conditions of Theorem 11.1 are satisfied and* $f(t)$ *is bounded in* $(-\infty, +\infty)$. *Then system (1.1) has a unique solution* $x^0(t)$ *bounded in* $(-\infty, +\infty)$ *(limit motion). This solution is exponentially stable in the large, i.e., for any solution* $x(t)$ *and any* $t > t_0$, *(11.3) holds, where* $\gamma > 0$, $\varepsilon > 0$ *are determined only by the parameters of the linear part of the system and the numbers* $\mu_1, \ldots, \mu_\varkappa$. *If* $f(t)$ *is an almost-periodic or T-periodic function vector, the solution* $x^0(t)$ *will also be, respectively, almost-periodic or T-periodic.*

We recall that for $\varkappa = 2$ the existence conditions for the diagonal matrix τ_d figuring in Theorems 11.1 and 11.2 are given in § 8. For $\varkappa = 1$, Tsypkin and Naumov /16/ have given methods for checking the frequency condition (8.3) which are convenient for engineering calculations, as well as methods for synthesis of stabilizing devices which guarantee absolute stability of the limit motion.

§ 12. Nonlinear sampled-data systems

A large class of nonlinear pulse systems is described, as shown in /53/, by difference equations which, in the previous notation, have the form

$$x_{t+1} = Px_t + q\varphi(\sigma_t), \quad \sigma_t = r^* x_t. \tag{12.1}$$

We restrict ourselves to the case of a single nonlinearity, $\varkappa = 1$. The transfer function from the input φ to the output $(-\sigma)$ will be $\chi(e^\lambda)$, where $\chi(\lambda)$ is determined as before by (1.5).

We shall say that the principle case obtains if all the roots of equation (1.3) lie inside the unit circle, and the simple critical case if one of the roots of this equation is $\lambda = 1$ and the remaining roots lie inside the unit circle.

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* Our proof is that of /31/, where the case $\varkappa = 1$ is considered.

Tsypkin /14/ obtained the following results.[*]

Theorem 12.1. *Suppose that the principal case obtains and that the function* $\varphi(\sigma)$ *satisfies (0.4).*

(I) If for all ω, $0 \leqslant \omega \leqslant 2\pi$

$$\mu_0^{-1} + \operatorname{Re}\chi(e^{i\omega}) > 0, \tag{12.2}$$

then system (12.1) is exponentially absolutely stable: there exist constants $\varepsilon > 0$, $\tau > 0$ *depending only on* P, q, r, μ_0 *such that for any solution* $x(t)$

$$\mu_0^{-1} + \operatorname{Re}\chi(e^{i\omega}) > 0. \tag{12.3}$$

(II) If the rank of each of the matrices (6.2) is ν *and (12.2) holds with the sign* $>$ *replaced by* $>$, *then the solution* $x_t \equiv 0$ *of system (12.1) is asymptotically stable in the large.*

(III) Part (I) remains valid if in (12.1) $\varphi(\sigma_t)$ *is an arbitrary function of* x_t *and* t *(and, possibly, of certain parameters), related to* σ_t *by (0.4).*

Kalman and Szegö /36/ have shown that when (II) is satisfied, system (12.1) has a Lyapunov function of type (2.2) such that $V_{t+1} \leqslant V_t$. Using a Lyapunov function of class **L** (see (2.11)), Szegö obtained /37/ a frequency condition more general in form than (12.3):

Theorem 12.2 (Szegö). *Consider system (12.1) in the principal case, and suppose that each of the matrices (6.2) has rank* ν, $\varphi(\sigma)$ *is a differentiable function satisfying (0.4), and* $0 < \varphi'(\sigma) < \alpha$. *If for some real numbers* β *and* γ *and all* $\omega \in [0, 2\pi]$

$$\begin{gathered} \beta + \gamma > 0, \ (\beta + \gamma)\mu_0^{-1} - 2\beta r^* q - 2\alpha\beta (r^* q)^2 \operatorname{sign}\beta > 0, \\ (\beta + \gamma)\mu_0^{-1} + \operatorname{Re}[(\gamma + \beta e^{i\omega})\chi(e^{i\omega})] - \frac{\alpha|\beta|}{2}|(e^{i\omega} - 1)\chi(i\omega)|^2 > 0, \end{gathered} \tag{12.4}$$

then the solution $x_t \equiv 0$ *of system (12.1) is asymptotically stable in the large.*

Under the assumptions of Theorem 12.2, Dmitriev has obtained certain other, less restrictive, conditions for stability in the large. His result is based on the use of a Lyapunov function of type (9.1).

§ 13. Survey of the basic frequency conditions

Consider systems (1.1) and (1.2) in the principle case, for $\varkappa = 1$. The following three frequency conditions were basic in almost all the theorems[**] of §§ 6—7:

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[*] In /14/, instead of the estimate (12.3) (see below) only asymptotic stability is affirmed, but the theorem is proved for systems more general in form than (12.1). The present formulation is due to Yu. A. Dmitriev.

[**] Note that we have not formulated the additional conditions, as a rule only slightly restrictive, which must be satisfied to ensure the validity of the following statements (see the corresponding theorems of §§ 6—12). In many cases the signs $>$ in (13.1)—(13.3) may be replaced by $>$, and θ may be $\pm\infty$. Condition (13.3) coincides with (9.4) for $\tau = \tau_1/\tau_2$.

$$\mu_0^{-1} + \operatorname{Re}\chi(i\omega) > 0, \tag{13.1}$$

$$[\mu_0^{-1} + \operatorname{Re}\chi(i\omega)] + \vartheta \operatorname{Re}[i\omega\chi(i\omega)] > 0, \tag{13.2}$$

$$\begin{gathered}[\mu_0^{-1} + \operatorname{Re}\chi(i\omega)] + \vartheta \operatorname{Re}[i\omega\chi(i\omega)] + \\ + \tau\omega^2 [1 + (\alpha_2 - \alpha_1) \operatorname{Re}\chi(i\omega) - \alpha_1\alpha_2 |\chi(i\omega)|^2] > 0.\end{gathered} \tag{13.3}$$

Inequality (13.2) must be satisfied for some ϑ, and inequality (13.3) for some ϑ and $\tau > 0$. In §§ 4 and 9 geometrical constructions for determining the existence of the parameters ϑ and τ were described.

Condition (13.1) is the most restrictive. In this case the linear part of the system has high stabilizing properties: the closed system (1.2) will be exponentially stable in the large for a wide class of nonlinearities; any function of x and t (in particular, a hysteresis function) which is related to the input $\varphi = \varphi(t)$ by the relation $0 \leqslant \sigma\varphi \leqslant \mu_0\sigma^2$ may be the input $\sigma = \sigma(t)$ of the nonlinear block. If the nonlinearity is of the backlash type and the condition $0 \leqslant \dot{\sigma}\dot{\varphi} \leqslant \mu_0\dot{\sigma}^2$ is satisfied (for the backlash $\mu_0 = 1$), then any solution $x(t)$ of the system (1.2) tends when $t \to \infty$ to some stationary state. If the system is subjected to a T-periodic, almost periodic or bounded in time external action, then the system has an asymptotically (exponentially) stable-in-the-large T-periodic, almost-periodic, or simply bounded limit motion, respectively (§ 11).

A linear unit for which Popov's less restrictive frequency condition (13.2) is satisfied, possesses less stabilizing properties. The closed system (1.2) is absolutely stable in the class of any continuous or discontinuous nonlinearities $\varphi = \varphi(\sigma)$ satisfying condition (0.4). If the nonlinear unit is nonstationary ($\varphi = \varphi(\sigma, t)$) or hysteresis ($\varphi = \varphi[\sigma, \varphi_0]_t$), then absolute stability is maintained only under certain additional conditions (monotone variation of the function $\varphi(\sigma, t)$ for fixed t, definite circulation of the hysteresis loop of $y\varphi[\sigma, \varphi_0]_t$ —see § 8). A system with T-periodic, almost-periodic or bounded external load may not have a single limit motion; it remains dissipative.

A linear unit satisfying the least restrictive frequency condition (13.3) also possesses the weakest stabilizing properties. A closed system is absolutely stable for differentiable nonlinearities $\varphi = \varphi(\sigma)$ satisfying, in addition to condition (0.4), also condition (9.2). A system with bounded (in particular, periodic or almost-periodic) external action remains dissipative.

When conditions (13.1), or (13.3) are satisfied, system (1.2) has a Lyapunov function of the respective forms

$$V = x^*Hx,$$

$$V = x^*Hx + \vartheta\int_0^\sigma \varphi(\sigma)\,d\sigma,$$

$$V = x^*Hx + 2x^*h\varphi(\sigma) + \gamma\varphi(\sigma)^2 + \vartheta\int_0^\sigma \varphi(\sigma)\,d\sigma.$$

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CONTINUUM PROBLEMS OF ANALYTICAL MECHANICS

The aim of the present paper is to analyze research in analytical mechanics pertaining to the field which unifies the mechanics of discrete and continuous systems, and to formulate some problems of analytical mechanics relating essentially to continuum mechanics.

We recall that the mechanics of discrete and continuous systems are unified mainly by the physical representation of the field of mechanico-geometrical objects and by the mathematical description of physical phenomena occurring in the field. These descriptions generally employ non-Riemannian geometry, and, in particular, variational principles, which are similar in form in the mechanics of discontinuous and continuous systems.

We emphasize that it is now almost impossible to separate physical concepts from their mathematical description.

The bibliography includes sources which have contributed to the formation of the ideas of the present report, irrespective of the references in the text of the paper.

1. The basis for introducing the geometrical representations connected with the equations of motion of bound systems is provided by generalized coordinates and quasi-coordinates.

It is well-known that the properties of a space are determined by the method whereby its points are arithmetized by selected coordinates, metric and affine-connection coefficients, which establish the equality of tensor quantities in neighboring points of the space. These three elements used in describing the properties of a space are in general independent. The metric of the configuration space or, in general, of a continuous medium is usually determined on the basis of the expression of the kinetic energy of an element of the medium.

Let us recall this relation. Consider an element of a continuous medium. We assume temporarily that it consists of N particles. The kinetic energy of this system, even in the presence of nonstationary relations, may be represented in the form

$$2T = \sum_{j=1}^{N} a_{ik}^{(j)} \dot{x}_{(j)}^{i} \dot{x}_{(j)}^{k}, \quad (i,\ k = 1,\ 2,\ 3,\ 4), \tag{1.1}$$

where $\dot{x}^{\mu}_{(j)}$ are the generalized velocities of the j-th particle, and $a^{(j)}_{\mu\nu}$ the corresponding coefficients of mobility.

It is assumed that the coordinate $x^{4}_{(j)}$ is a linear function of the time t. Putting

$$x^{\mu}_{(j)} = x^{\mu}_{c} + \xi^{\mu}_{(j)};\ \ \xi^{4}_{(j)} = 0;\ \ \sum_{j=1}^{N} \rho^{(j)} \xi^{\mu}_{(j)} = 0, \tag{1.2}$$

where $\rho^{(j)}$ is the mass of the j-th particle, we obtain from (1.1)

$$2T = \dot{x}^i_c \dot{x}^k_c \sum_{j=1}^{N} a^{(j)}_{ik} + \sum_{j=1}^{N} a^{(j)}_{ik} \dot{\xi}^i_{(j)} \dot{\xi}^k_{(j)} + 2\dot{x}^i_c \sum_{j=1}^{N} a^{(j)}_{ik} \dot{\xi}^k_{(j)}. \tag{1.3}$$

Passing again to a continuous medium, we assume that $a^{(j)}_{\mu\nu}$ and $\rho^{(j)}$ are differentiable functions of the coordinates x^μ_c. We put:

$$g_{\mu\nu} = \lim_{\xi^\mu_{(j)} \to 0} a^{(j)}_{\mu\nu} : \rho^{(j)}. \tag{1.4}$$

Then

$$a^{(j)}_{\mu\nu} = \rho^{(j)} \left[g_{\mu\nu} + \frac{\partial}{\partial x^\lambda_c} \left(\frac{a^{(j)}_{\mu\nu}}{\rho^{(j)}} \right) \xi^\lambda_{(j)} + \cdots \right]. \tag{1.5}$$

We also put

$$\frac{1}{\Delta V} \sum_{j=1}^{N} \rho^{(j)} = \rho; \quad \frac{1}{\Delta V} \sum_{j=1}^{N} \rho^{(j)} \dot{\xi}^\mu_{(j)} \dot{\xi}^\nu_{(j)} = -\tau^{\mu\nu}, \tag{1.6}$$

where ΔV is the volume element.

The kinetic energy $2T$ then assumes the form:

$$2T = g_{ik} (\rho \dot{x}^i_c \dot{x}^k_c - \tau^{ik}) \Delta V + O(\xi^\lambda_{(j)}), \tag{1.7}$$

where $O(\xi^\lambda_{(j)}) \to 0$ when $\xi^\lambda_{(j)} \to 0$.

Contracting the medium element into a point, while keeping the right-hand sides of (1.6) finite, we find

$$2K = g_{ik} (\rho \dot{x}^i \dot{x}^k - \tau^{ik}), \tag{1.8}$$

where K is the kinetic energy density of the element of the continuous medium. The tensor $g_{\mu\nu}$ is usually regarded as the metric tensor. The expressions in parentheses in the right-hand side of (1.8) are one of the forms of the components of the energy-momentum tensor, or, following Levi-Civita, of the kinetic stress tensor. Consequently, an intrinsic relation exists between the metric of the space and the energy-momentum tensor. This is considered in more detail below.

The coefficients of affine connection do not follow from consideration of the kinetic energy. To determine them we must use the differential equations of motion. We mention, for example, the equations /10/:

$$\frac{dv^a}{dt} + \Gamma^a_{bc} v^b v^c = X^a, \quad (a,\ b,\ c = l+1,\ l+2, \ldots,\ N,\ N+1), \tag{1.9a}$$

$$\Gamma^a_{bc} v^b v^c = X^a, \quad (a = 1,\ 2, \ldots, l;\ b,\ c = l+1, \ldots, N+1), \tag{1.9b}$$

$$v^a = \beta^a_j \dot{q}^j, \quad (a,\ j = 1,\ 2, \ldots,\ N,\ N+1), \tag{1.9c}$$

$$\Gamma^j_{ik} \dot{q}^i \dot{q}^k = Q^j, \quad (j = N+2, \ldots, 3n). \tag{1.9d}$$

$$X^a = \beta^a_j Q^j, \quad Q^j = \alpha^j_a X_a, \quad (a,\ j = 1,\ 2, \ldots, N+1). \tag{1.9e}$$

Equations (1.9a) and (1.9c), due to I. Schouten, determine the law of motion of a system with geometric and nonholonomic connections. The remaining equations express the reactions of the connections. It is important to note that the coefficients Γ^a_{bc} of affine connection, generalizing the Christoffel symbols of the second kind, are determined from the equation of motion with an accuracy to within a certain term antisymmetric in the indexes b and c. These coefficients comprise one of the fields of mechanico-geometrical objects mentioned above. The facts we have recalled are the basis of extensive research extending the possibilities of geometrical interpretation of the motions of discrete and continuous mechanical systems. In particular, this has led to the creation of modern nonholonomic geometry, in the well-known monograph of I. A. Schouten and D. G. Struik /27/, in numerous papers of G. Vränceanu, of whose recent papers we mention /42/, Z. Horák, A. Wundheiler and others. These investigations are concerned essentially with a branch of non-Euclidian geometry, and are beyond the scope of the present review.

One of the founders of the modern geometry of nonholonomic systems is V. V. Vagner. His works have contributed, on one hand, to the development of a geometrical theory of nonholonomic manifolds, and, on the other hand, have indicated the possible applications of this theory to mechanical problems, in particular, of the dynamics of a rigid body /2—4/. Vagner considers the configuration space of a rigid body with a fixed point, moving by inertia (Euler space), and proves that this space is Riemannian. Then, imposing a nonholonomic constraint on the body, he sets up the equations of motion of the representative point in a nonholonomic space imbedded in the Euler space; finally, using the geometrical properties of the nonholonomic space, he obtains the solution of the problem in terms of hypergeometrical functions of time /3/. Nevertheless, these papers do not provide a general technique for effective application of the geometrization of dynamic problems of nonholonomic systems to the solution of concrete problems.

J. Synge writes in the monograph /26/ that, in principle, geometrical methods, in particular tensor analysis methods, cannot be regarded as a means for effective solution of special problems, and can be used only for describing the most general laws of motion of mechanical systems. Later a number of papers, of which we mention that of V.V Dobronravov /6/, weakened Synge's affirmation by generalizing a number of fundamental theorems of the dynamics of holonomic systems to the dynamics of nonholonomic systems, these results being based on substantial utilization of the apparatus of tensor analysis. We mention that among these theorems are some which make it possible to find the integrals of differential equations of motion /6, 14/. We touch upon this problem again in the sequel.

Both recent and previous investigations in the geometry of the dynamics of discrete systems link the geometrical investigations of G Vränceanu /42/ with the geometrical interpretation of motions of nonholonomic systems, due to E. Clauser /31/, M. Haimovici /34/, and J. Klein /36/. The last-mentioned paper differs from the others by its use of the metric of Finsler spaces to establish a generalized principle of least curvature.

To summarize, it can be said that the greatest progress in this field is bound up with geometrical interpretations of the general laws of motion of material systems. Basically, these interpretations are intrinsically related to new methods for solving complicated problems of mechanics, but development of effective methods for solving concrete problems on the basis of generalized geometrical representations is subject to further investigations. It is essential for our present purpose that geometrization of the laws of motion of discrete systems is one of the links between the mechanics of discrete and continuous systems.

Problems of geometrical generalizations are also related to "unified field theory." Among the numerous papers in this direction we mention several which develop non-Riemannian geometry, for example V. Hlavaty /35/, in which the determination of the coefficients of affine connection is studied, and of T. Takasu /40, 41/, which apply the methods of nonholonomic geometry to the extended general theory of relativity.

2. Approximate methods for solving problems of continuum mechanics lead to the formulation of new problems of the mechanics of systems with finitely many degrees of freedom. As an example we mention the investigations of I. V. Nikolenko on the statics of thin beams /18, 19/. These papers continue to develop the analogy between the motion of a rigid body and the equilibrium modes of thin beams, pointed out by Kirchhoff. Nikolenko considers problems concerning the equilibrium of constrained thin beams with nonlinear constraints which are nonintegrable in particular cases. The author uses methods introduced by Vagner in problems of the motion of a rigid body, with appropriate changes. The problems considered by Nikolenko refer to a narrow class of problems of continuum mechanics and do not therefore yield generalizing conclusions unifying the mechanics of discrete and continuous systems. An outline of the possible extension of the methods of classical analytical mechanics to the mechanics of the continuum is given in the book of J. Leech /14/ and others. The generalized coordinates of the system are replaced by field variables, and the three (in the general case) spatial coordinates and time are regarded as independent variables.

The Ostrogradskii-Hamilton principle is used to construct systems of differential equations of motion of the continuum elements. But Leech points out that there is no rigorous foundation for the application of this principle to problems of continuum mechanics (see /14/, pp. 119 and 120).

Let us give some illustrative example. In the applied mechanics of deformable bodies, approximate representation of the required quantities by various expansions is customary. The coefficients of these expansions are special forms of field variables. Of course, if these coefficients depend on time, they may be called generalized coordinates. However, this term is also regarded in more general cases as the equivalent of field variables.

Let us first recall some well-known facts from the theory of shells. One of the old methods for reducing three-dimensional problems of elasticity theory to two-dimensional problems is based on the expansion of the quantities determining the stress-strain state of an element of the shell in series of ascending powers of the distance of the element from the middle surface.

The coefficients of these expansions, which are functions of the coordinates of the middle surface, must be regarded as field variables. By using the equations of motion of an element of a continuous medium, the boundary-value problem of the theory of shells can be reduced to the determination of six field variables /9/. In the absence of body forces the kinematic boundary conditions generate the equations of the constraints. It is clear that these constraints are not geometrical or integrable. In the presence of body forces we do not obtain equations of constraints in the classical sense at all. These facts confirm the expediency of investigations aimed at justifying the use of the Ostrogradskii-Hamilton principle in continuum mechanics, in particular in the theory of shells.

Let us consider a problem of analytical mechanics arising in connection with the approximate solution of a hydroelasticity problem, concerned with the motion of a shell carrying a nonviscous liquid with a free surface in a conservative force field. This example is only of heuristic value.

Not solving the hydroelastic problem completely, let us try to determine the parameters which will enable us to find the position of the shell and the hydrodynamic pressure without investigating the field of the relative velocities of the liquid elements. First let us consider the general formulation of an approximate solution.

With the nondeformed shell we associate a moving Cartesian coordinate system $O\xi_1\xi_2\xi_3$. The required quantities are approximated by polynomials of the form

$$f = \sum_{\alpha, \beta, \gamma = 0}^{m} a_{\alpha\beta\gamma}\xi_1^{\alpha}\xi_2^{\beta}\xi_3^{\gamma}. \tag{2.1}$$

The coefficients a $a_{\alpha\beta\gamma}$ are unknown functions of time, to be determined. Their mechanical meaning is characterized by the physical quantity represented by (2.1). They can be regarded as field variables or, in the more customary terminology, as generalized coordinates of the system, if the displacements of the points of the shell and the hydrodynamic pressure are approximately represented by the form (2.1), or as generalized velocities, if the components of the relative velocity vector of the liquid element are approximately represented thereby. The kinematic conditions imposed on field variables in this case follow from the condition of incompressibility, which is approximately replaced by the condition of constant liquid volume

$$V = V_0, \tag{2.2}$$

from the condition at the wet surface of the shell

$$v_{rn} = \left(\frac{\partial \vec{u}}{\partial t}\right)_n \tag{2.3}$$

(where $\mathbf{v}_r$ is the relative velocity vector of the liquid element and $\mathbf{u}$ is the displacement vector of the shell element) and from the condition

$$p(t, \xi_1, \xi_2, \xi_3) - p_0 = 0, \tag{2.4}$$

at the free surface S (where p is the hydrodynamic pressure and p_0 the pressure at the free surface).

Condition (2.4) implies

$$\frac{\partial p}{\partial t} + \mathbf{v}_r \cdot \operatorname{grad} p = 0 \text{ (at } S). \tag{2.5}$$

Relation (2.5) and the approximate representations of the required functions in the form (2.1) lead to the equation of a nonintegrable constraint. In particular cases when a moving coordinate system can be chosen (the possibility of such a choice is ascertained by visual observations of the motion of the pendulum carrying the liquid), such that the product $\mathbf{v}_r \cdot \operatorname{grad} p$ is sufficiently small, we find from condition (2.5)

$$\left(\frac{\partial p}{\partial t}\right)_S = 0 \tag{2.6a}$$

or, approximately

$$\sum_{\alpha, \beta, \gamma=0}^{m} \dot{p}_{\alpha\beta\gamma}\xi_1^{\alpha}\xi_2^{\beta}\xi_3^{\gamma} = 0. \tag{2.6b}$$

Here and below the dot denotes differentiation with respect to time. We have again obtained the equation of a nonintegrable constraint, but simpler than that following from (2.5).

The presence of a nonholonomic constraint (2.6b) precludes use of the Ostrogradskii-Hamilton and Euler-Lagrange principles for an approximate solution of the problem in question, unless there exists an integrating factor for the left-hand side of (2.6b).

It seems that the applicability conditions of these principles to hydroelasticity problems are worthy of further study, since the approximate method described here is nevertheless intrinsically related to the precise methods for solving these problems.

If the integral characteristics of the motion of the "shell-liquid" system, such as the frequency of natural oscillations of the system, are to be determined, condition (2.6b) may be averaged over the region in which the free surface is moving, if the conditions of the problem provide an approximate delineation of this region.

In this case, after averaging, condition (2.6b) yields the equations of an integrable constraint. The physical meaning of this approach is fairly clear.

Instead of the motion of a body with a cavity partially filled with liquid having a mobile free surface, let us consider the motion of a body carrying a liquid with a quasi-fixed free surface. It is clear that such an approximate approach reduces the number of degrees of freedom of the system with respect to the generalized coordinates and distorts the spectrum of free oscillation frequencies. Since this simplified solution of the problem may nevertheless be useful, we shall consider it briefly.

Since in the simplified formulation the constraints are stationary and integrable, the Euler-Lagrange principle may be used instead of the more customary Ostrogradskii-Hamilton principle. This makes it possible to use the energy integral to determine the kinetic energy of the liquid. We find

$$T^{(ж)} = T_0^{(ж)} - \Pi^{(ж)} - \iint\limits_{(\sigma)} \left[\int\limits_0^t (p - p_0)\, v_{\sigma n}\, dt\right] d\sigma, \tag{2.7}$$

where $T^{(ж)}$ is the kinetic energy of the liquid, $T_0^{(ж)}$ is the initial kinetic energy of the liquid, $\Pi^{(ж)}$ is the potential energy of the liquid, σ is the wet surface of the shell, $\mathbf{v}_n$ is the absolute velocity of the liquid element. Equation (2.7) enables us to exclude the field of relative velocities of the liquid elements from our considerations.

The kinetic energy of the "shell-liquid" system can be represented in the form

$$T = T_1(q_i, \dot{q}_i) + \int_0^t T_2(q_i, q_{N+j}, \dot{q}_i)\,dt, \quad (i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, l). \tag{2.8}$$

where T_1 is a quadratic form in the generalized velocities and T_2 depends linearly on the generalized velocities. The generalized coordinates $q_1, \ldots, q_N$ determine the motion of the coordinate system $O\xi_1\xi_2\xi_3$ and the relative displacements of the shell elements. The independent generalized coordinates $q_{N+1}, \ldots, q_{N+l}$ are equal to the coefficients $a_{\alpha\beta\gamma}$ in the approximate expression for the hydrodynamic pressure.

Eliminating the time t as an independent variable on the basis of the energy integral, choosing a new independent variable q_1 and putting

$$\xi = \int_{q_{10}}^{q_1} T_2(q_i, q_{N+j}, q_i')\,dq_1, \tag{2.9}$$

where the prime denotes differentiation with respect to q_1, we obtain a variational problem

$$\Delta \int_{q_{10}}^{q_{11}} 2(h - \Pi)\sqrt{\frac{T_1(q_i, q_i')}{h - \Pi - \xi}}\,dq_1 = 0, \tag{2.10}$$

where Δ denotes asynchronous variation, and h is the energy constant.

Introducing the notation:

$$F(q_i, q_{N+j}, q_i', \xi) = (h - \Pi)\sqrt{\frac{T_1(q_i, q_i')}{h - \Pi - \xi}}, \tag{2.11}$$

we find from (2.10)

$$\frac{d}{dq_1}\left(\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial \xi}\frac{\partial T_2}{\partial q_i'} - \frac{d}{dq_1}\frac{\partial F}{\partial q_i'}\right) - \frac{\partial F}{\partial \xi}\left(\frac{\partial T_2}{\partial q_i} - \frac{d}{\partial q_1}\frac{\partial T_2}{\partial q_i'}\right) = 0, \tag{2.12a}$$

$$\frac{d}{dq_1}\frac{\partial F}{\partial q_{N+j}} - \frac{\partial F}{\partial \xi}\frac{\partial T_2}{\partial q_{N+j}} = 0, \quad (i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, l). \tag{2.12b}$$

Equations (2.12a)—(2.12b) are generalizations of the well-known Jacobi equations.

Equations (2.12a)—(2.12b) must be considered with additional natural initial conditions:

$$\left.\frac{\partial T_2}{\partial q_i'}\right|_{q_1 = q_{10}} = 0, \quad (i = 2, \ldots, N). \tag{2.13}$$

The system (2.12a)—(2.12b) is of order $(N-1)+l$. The number of initial conditions, including (2.13), is equal to the order of the system.

These examples illustrate the trend of the possible extensions of the foundations of classical analytical mechanics originating in problems of continuum mechanics.

3. We turn to the problems of analytical mechanics of the continuum which we believe to be the most important. We shall concentrate attention on two problems.

First we shall consider the relation between the kinetic-stress functions which we introduced in 1953—1954, and a variational principle analogous to the Ostrogradskii-Hamilton principle. Then we shall discuss the relation between this principle and the basic law of continuum mechanics, which establishes the relationship between the kinetic-stress tensor and the metric of the deformed space, in particular, the strain tensor. Finally, we shall touch upon the general problem of integrating the equations of continuum mechanics using extensions of the theorems of classical analytical mechanics.

We shall use the mathematical apparatus of the general theory of relativity. This is perfectly natural, since one of the phenomenological foundations of the general theory of relativity is classical continuum mechanics. We emphasize that the physical principles of the general theory of relativity will not be discussed.

a) It is well-known that the system of equations of motion of an element of a continuous medium and the equation of continuity can be represented in the form of an invariant equation

$$\nabla_i T^{ik} = \rho F^k, \quad F^4 = 0, \quad (i, k = 1, 2, 3, 4), \tag{3.1}$$

where T^{ik} are the components of the kinetic-stress tensor figuring in relation (1.8).

Consider the case when body forces are absent; let us temporarily refrain from postulating the Euclidean properties of the space. Equations (3.1) are then identically satisfied by the tensor

$$P^{ik} = \varkappa\left(R^{ik} - \frac{1}{2} g^{ik} R\right), \tag{3.2}$$

where R^{ik} are the components of the contracted curvature tensor, g^{ik} are the contravariant components of the matric tensor, and $\varkappa$ is an arbitrary constant /29/. In order to set up the tensor of kinetic-stress functions, which satisfies the system (3.1) of homogeneous equations identically in Euclidean space, we use a method of metric variation which consists in giving the components g_{ik} of the metric tensor increments $\varepsilon\Phi_{ik}$, where ε is a small parameter. Putting $\varkappa = \varepsilon^{-1}$ and expanding the right-hand side of (3.2) in ascending powers of ε, we find

$$P^{ik} = Q^{ik}_{(1)} + \varepsilon Q^{ik}_{(2)} + \cdots. \tag{3.3}$$

It may be proved that the $Q^{ik}_{(1)}$ identically satisfy equations (3.1) in the Euclidean metric in the absence of body forces, and consequently form the tensor of kinetic-stress functions /9, 10/.

According to theorems in Eddington /29/, the tensor P^{ik} is the functional derivative of the scalar $\varkappa R$

$$P^{ik} = \frac{\delta(\varkappa R)}{\delta g_{ik}}. \tag{3.4}$$

The kinetic-stress functions are obtained by letting $\varepsilon \to 0$ in the functional derivatives (3.4).

The functional generating the derivatives (3.4) has the form

$$I = \varkappa \int\limits_{(\Omega)} R \sqrt{g}\, d\Omega, \tag{3.5}$$

where Ω is a domain in the four-dimensional space-time manifold.

Equation (3.4) is related to the condition

$$\delta I = 0, \tag{3.6}$$

which is formally similar to the Ostrogradskii-Hamilton principle.

We recall that the main difference between this principle and the Ostrogradskii-Hamilton principle is that in determining δI the coordinates x^i $(i = 1, 2, 3, 4)$ are varied, whereas when applying the Ostrogradskii-Hamilton principle they remain constant. Equality (3.6) results from the invariance of R under point transformations of the coordinates /29/.

We note that this method yields ten kinetic-stress functions, corresponding to the number of quantities determining the energy-momentum tensor. However, upon consideration of the geometrical meaning of these functions we may assert that the number of functions essential for the solution of problems is four, since, without loss of generality, four-dimensional space may be provided with an orthogonal coordinate system and a metric tensor having four essentially different components. These geometrical considerations lead directly to the conclusion that in problems of statics the number of essentially different stress functions is at most three. In two-dimensional dynamical problems of the theory of elasticity, for example in problems of shell mechanics, the number of these functions is also three.

The paper of I.S. Arzhanykh /1/ stimulated the development of the theory of kinetic-stress functions. Our kinetic-stress functions have been used by S. Kaliski (Poland) and V.N. Ionov in the solution of three-dimensional dynamical problems of the theory of elasticity.

b) As already indicated, extension of methods of analytical mechanics to the mechanics of a continuous medium depends largely on selection of the field variables or thermomechanical parameters of state. For example, L.I. Sedov and M.E. Eglit regard as field variables the components of the finite strain tensor D_{ik}, those of the stress tensor σ^{ik}, the temperature T and certain other quantities depending on the specific model of the continuous medium.

When this is done, there arise analytical problems with nonholonomic relations between these parameters, which enable us to indicate new general mathematical descriptions of the processes taking place in various types of continuous media /24, 28/. Eglit has shown that this method can be used to describe, for example, the relaxation of stresses in some high polymers.

Other very general approaches to the choice of field variables can be found in the monograph of A.A. Il'yushin /7/ and in other literature on the mechanics of a continuous medium. A detailed analysis of these investigations is beyond the scope of the present report, but we shall refer to them in discussing the problems of continuum mechanics which we have selected for further analysis.

One of the basic problems of the mechanics of a rigid continuous medium is the functional relation between the kinetic-stress tensor of the energy-momentum tensor and the finite strain tensor. We note preliminarily the following.

As is well-known, classical mechanics of a continuous medium compares two states of matter: the initial, undeformed state and the deformed state. These states are characterized by their respective metrics. The components of the strain tensor are determined by the differences of the corresponding components of the metric tensors, defined in Lagrangian coordinate systems, in the deformed and undeformed media.

Due to the introduction of the strain tensor, its components must be included among the field variables instead of those of the metric tensor, although the latter is apparently more natural. The expediency of this replacement is confirmed, in our opinion, by the investigations of E. Kröner /12, 37/, J.D. Eshelby /30/ and others, in continuous dislocation theory. These authors were also forced to avoid unconditional postulation of the absence of internal curvature of the space connected with the deformable medium, and, consequently, to dispense with the St. Venant compatibility conditions. The author of the present report arrived at these conclusions in 1938 /8/.

We shall consider two versions of the theory, corresponding to the introduction as field variables: a) of the components of the metric tensor, which vary in the deformation process; b) of the components of the finite strain tensor. The basis of the theory is a variational relation related to the d'Alembert-Lagrange principle. Following the statements of J. Leech /14/ and the research on the analytical apparatus of the general theory of relativity of A. Eddington /29/, T. de Donder /33/ and others, who considered a modification of the Ostrogradskii-Hamilton principle, we postulate this variational relation. In the first version the field variables are the components of the metric tensor g_{ik}, the components of the kinetic-stress tensor T^{ik}, the components of the temperature gradient $\partial_i T$ and the temperature T. The tensor quantities enable us to construct a set of invariants, making use also of the physical tensor characteristics of the properties of the medium being studied.

Some indeterminacy arises here: only in exceptional cases can it be eliminated by theoretical considerations; such is the case, for example, in the general theory of relatively. Otherwise we may obtain results which require experimental verification and may even be devoid of physical meaning.

To set up the equation which expresses the d'Alembert-Lagrange principle, we consider a special case. Let us find the elementary work performed by the action of force and temperature when the metric is varied by deformation of a four-dimensional element of the medium.

Let us assume that the variations δg_{ik} of the metric tensor are zero at the boundary of the four-dimensional region filled with the medium. We find

$$\delta A = \int_{(\Omega)} \{\delta K - \varkappa T^{ik}\delta g_{ik} + A^{ik}(T - T_0)\,\delta g_{ik} + \\ + B^{\alpha\beta.\,ik}\partial_\alpha T\partial_\beta T\delta g_{ik} + \dots\}\, \sqrt{g}\, d\Omega = 0, \qquad (3.7)$$

where K is an invariant depending on the g_{ik} and their derivatives with respect to the coordinates x^i $(i = 1, 2, 3, 4)$, T_0 is the initial temperature, $\varkappa$, A^{ik}, $B^{\alpha\beta.\,ik}$ are the physical characteristics of the thermomechanical properties of the material.

With the aid of transformations indicated, for example, in /29/ the variational relation (3.7) can be given the following form:

$$\int_{(\Omega)} \{P^{ik} - \varkappa T^{ik} + A^{ik}(T - T_0) + B^{\alpha\beta.\,ik}\partial_\alpha T\partial_\beta T + \dots\}\, \delta g_{ik} \sqrt{g}\, d\Omega = 0. \qquad (3.8)$$

Hence we find

$$\varkappa T^{ik} = P^{ik} + A^{ik}(T - T_0) + B^{\alpha\beta.\,ik}\partial_\alpha T\partial_\beta T + \dots \qquad (3.9)$$

The first term on the right-hand side of (3.9) represents the dependence of the kinetic-stress tensor on the varying intrinsic metric of the deformable space associated with the continuous medium, i.e., on its deformation; the remaining terms have a thermomechanical interpretation. It is clear that there always exists a tensor A^{ik} corresponding to the addition of terms in Hooke's law depending on thermal expansion /17, 38, 39/.

The term $B^{\alpha\beta.\,ik}\partial_\alpha T\partial_\beta T$ allows for the flux of thermal energy, and also the increase in entropy /21/.

According to well-known theorems the tensor P^{ik} satisfies the condition of energy-momentum conservation /29/. Postulating that the energy-momentum tensor obeys the conservation conditions, we define, with an accuracy to within, say, a scalar factor, the components of the generalized energy-momentum tensor ϑ^{ik} by the expression

$$\vartheta^{ik} = \varkappa T^{ik} - A^{ik}T - B^{\alpha\beta.\,ik}\partial_\alpha T\partial_\beta T + \dots. \qquad (3.10)$$

From (3.9) we then find

$$\vartheta^{ik} = P^{ik}. \qquad (3.11)$$

Formula (3.9) or (3.11) expresses the required relation between the kinetic stress tensor and the metric tensor. It is simple to introduce in (3.9) and (3.11) the strain tensor

$$g_{ik} = D_{ik} + g_{ik}^{(0)} \qquad (3.12)$$

(where $g_{ik}^{(0)}$ are the initial values of the components of the metric tensor) and thus to arrive at the second version of describing the stress-strain state of a continuous medium. Introduction of the tensor D_{ik} corresponds to the special assumption that the variation $\delta g_{ik}^{(0)}$ vanish.

More general cases are considered in our paper /8/.

If we put

$$K=R \tag{3.13}$$

in (3.7) (where R is the contracted Riemann-Christoffel tensor), the tensor P^{ik} assumes the form (3.4), and (3.11) reduces to the Lagrange equations of the second kind

$$\frac{\delta R}{\delta g_{ik}}=\frac{\partial R}{\partial g_{ik}}-\sum_{\alpha=1}^{4}\frac{\partial}{\partial x^{\alpha}}\frac{\partial R}{\partial g_{ik,\alpha}}=Э^{ik}, \quad (i.\ k=1,\ 2,\ 3,\ 4), \tag{3.14}$$

where

$$g_{ik,\alpha}=\partial_{\alpha}g_{ik}. \tag{3.15}$$

We recall that equations (3.11) and (3.14) follow from the variational relation (3.8) for independent δg_{ik}.

If the components $Э^{ik}$ are zero, the variational relation (3.7) becomes a variational principle similar to the Ostrogradskii-Hamilton principle, encountered in the analytical apparatus of general relativity theory.

By variation with respect to other variable and completing the integrand of (3.7), we obtain equations figuring, for example, in the monographs of L.I. Sedov /4/ and I.I. Gol'denblat /5/. In this case nonholonmic relations between the field variables considered by Sedov and Eglit /28/ can be introduced, with appropriate alterations of the equations known from analytical mechanics. These alterations and their geometrical interpretation were mentioned above.

Deformation of a continuous medium can also be regarded as the result of some nonholonomic transformation. One should not then assume that the quantities $\delta g_{ik}^{(0)}$ vanish, but rather relate δg_{ik} to $\delta g_{ik}^{(0)}$ by the following relation, indicated in our paper /8/:

$$\delta g_{ik}^{(0)}=B_{ik}^{\cdot\cdot\alpha\beta}\delta g_{\alpha\beta}. \tag{3.16}$$

When this is done the analytical form of the conservation condition for the energy-momentum tensor $Э^{ik}$ changes.

Returning to (3.9) and (3.11), which yield the required relation between the kinetic stress tensor and the strain tensor, we make some brief remarks on the history of the problem.

The relation between Einstein's gravitational equations and Hooke's law was indicated in 1937 by Yu. Rumer /23/. In 1938 we investigated the relation between a general law of the form (3.9), without the introduction of temperature terms, Hooke's law, the extension of Hooke's law to the theory of small elastic-plastic deformations and the relationship between the stress tensor and the strain velocity in the mechanics of viscous fluids /8/. Difficulties connected with the contradiction between postulation of the Euclidean properties of space and the need to introduce the Riemann-Christoffel tensor, as well as the lack of information concerning the physics of rigid bodies, forced us to interrupt our investigations, although the role of the initial stresses, and other factors taken into consideration later by E. Kröner and others, were mentioned in the above paper.

E. Kröner, R. de Witt and others have pointed out anew the importance of the Einstein tensor P^{ik} for the mechanics of continuous media, and have given a physical interpretation of the internal curvature of a continuous deformable medium on the basis of dislocation theory /12, 30, 37/.

4. The above problems indicate the diversity of the problems raised by the extension of the methods of classical analytical mechanics to continuum mechanics.

Two mutually related problems are, in our opinion, particularly important: a) analysis of the conditions for applicability of classical variational principles in continuum mechanics; b) extension of the general methods for integration of the classical equations of analytical dynamics to the integration of the equations of continuum mechanics.

The latter problem, which is of particular interest, is touched upon, for example, in Gol'denblat /5/, J. Leech /14/ and K.A. Lur'e /16/.

When these problems are worked out, it will apparently be possible to indicate new, sound approaches to the solution of the basic problem of the mechanics of continuous media — the establishment of a general functional relation between the stress and strain tensors.

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PROBLEMS OF CONTROLLABILITY, OBSERVABILITY, AND STABILIZABILITY OF DYNAMIC SYSTEMS

1. The following three problems are of great importance in the theory of controllable dynamic systems, particularly in the theory of optimal processes.

1. The control problem (CP), i.e., the problem of calculating the forces which transfer the controlled object from a given state to another.
2. The observation problem (OP), i.e., the problem of the operation which determines the unknown coordinates of the object on the basis of observable quantities.
3. The stabilization problem (SP), i.e., the problem of creating a regulating action which ensures stable realization of a given motion on the basis of accessible information.

Each of these problems is often associated with demands for the best quality of the process in some sense or another; they then become part of the theory of optimal systems. However, CP, OP, and SP are also of independent interest; besides, even within the framework of optimality problems they display specific features, sometimes also difficulties. They are therefore worthy of special consideration. Well-defined methods for investigating these problems have now been developed and certain general regularities have been established. The purpose of this paper is to discuss some of these results.

The SP, including the problem of optimal stabilization, can be considered as a further development of the classical theory of Lyapunov stability /1, 2/. The theory of stabilization is closely bound up with the investigations of N.G. Chetaev and his students V.V. Rumyantsev and G.K. Pozharitskii on the stability of systems in the presence of mechanical forces of different natures. CP and OP have been intensely studied recently, particularly in connection with the theory of optimal systems. However, they are also closely related to the SP, playing an important role in its investigation. CP have been considered in the theory of optimal processes, on the basis of the maximum principles of L.S. Pontryagin and his school /3/, from the viewpoint of R. Bellman's theory of dynamic programming /4/, and within the framework of classical variational calculus. R. Kalman /5/ studies CP and OP for linear systems under conditions of minimum quadratic criterion functions; he formulates the duality between CP and OP and considers the canonical representation of linear systems. A significant role in the study of CP, OP, and SP is played by the investigation of specific controlled mechanical systems, where important results, such as the stabilization theory of complex gyroscopic systems developed by A. Yu. Ishlinskii /6/, have been obtained. We point out the relation between OP and problems of noise filtering and prediction /7/, which comes to light in the statistical development of the problem /8/, the relation between CP and the theory of invariant systems, and the relation between CP and OP and A.I. Lur'e's problem /9/ of absolute stability, in particular, with the aspect of this problem developed by V.M. Popov /10/. Many problems of control and observation figure in A.A. Fel'dbaum's theory of optimal systems /11/. A number of studies deal with numerical investigation and solution of applied CP and OP (see Ya.N. Roitenberg /12/). A satisfactory review of the huge literature on CP, OP and SP is beyond the scope of this paper. We have therefore mentioned only studies which define some basic direction of investigation and give some idea of its general character. For this reason we shall not refer to the numerous specific sources in the main text of the paper.

One possible approach to Problems 1 and 2 (and particularly to linear CP and OP) is to consider them as problems of constructing linear operators, satisfying given conditions, on the elements of some functional space. With this approach one can evaluate the problems from the general standpoint of functional analysis, and use, for instance, the extension theorem for linear operators /13/ in its specific forms, such as the moment problem, the problem of separation of sets, and so on. From this point of view, which we, in particular, have adopted in Sverdlovsk, we consider these problems in the present paper.

Problem 3, which develops stability theory, can be approached via the problem of synthesis of optimal systems, in particular the problem of analytical regulator design. A basic source of this point of view, which we adopt here, is the research of A.M. Letov /14/. These methods of SP investigation develop results of Lyapunov, Chetaev and others in the theory of stability of motion, in particular, the method of Lyapunov functions, modernized in accordance with the principles of dynamic programming.

2. Let us consider the control problem. Suppose a dynamic system is described by ordinary differential equations

$$\frac{dx}{dt} = f[t, x, u], \tag{2.1}$$

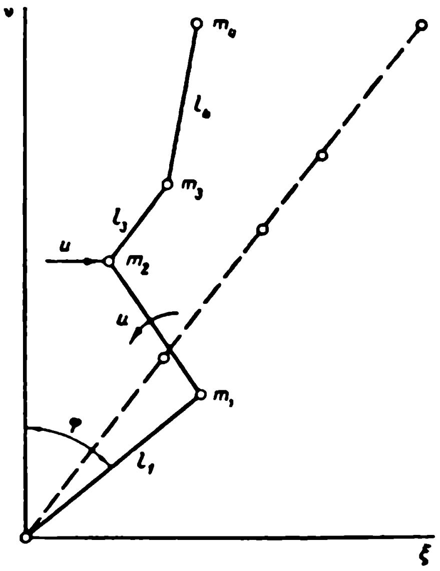
where x is the vector of phase coordinates x_i $(i = 1, \ldots, n)$ of the object (for example, $n = 2s$, $x_1, \ldots, x_n$ may be the generalized coordinates, and $x_{s+1}, \ldots, x_{2s}$ the generalized velocities, of a holonomic mechanical system) and u_j is the vector of control actions u_j $(j = 1, \ldots, r)$.

The coordinates u_j are either forces actually applied to the object, or quantities related in some way to the forces and introduced into (2.1) in a convenient form. Finally, the relation between the u_j and the x_i may be more general than (2.1).

Problem 2.1. Given the initial state $\{t^0, x^0\}$ of the object and the manifold of its final states $\{t^{(k)}, x^{(k)}\}$, described by parametric equations

$$x^{(k)} = s[z], \quad t^{(k)} = t^0 + \theta[z], \tag{2.2}$$

where z is a k-dimensional vector of parameters, restricted perhaps, by some condition which we denote symbolically by $z \in Z$. There may be additional restrictions and constraints.



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The problem is to determine values $z = z^0$ and the functions $u_j(t)$ $(t^0 \leqslant t \leqslant \leqslant t^0 + \theta)$ satisfying the given restrictions, such that for $u = u(t)$ a motion $x(t)$ starting at the point $x(t^0) = x^0$ reaches the point $x(t^{(k)}) = s[z^0]$.

The conditions determining the manifold M may preassign not only the final values of $x^{(k)}$, but also the values of $x(t^{(i)})$ at any other instant $t^{(i)}$ (including $t = t^0$), M being then determined by equations $x(t^{(i)}) = s^{(i)}[z]$, $t^{(i)} = t^0 + \theta^{(i)}[z]$, where $\{i\}$ is a finite or infinite index set. In this way we can, in particular, allow for restrictions on the motion coordinates $x(t)$. The formulation is similar if there are infinitely many degrees of freedom.

Example 2.1. Consider an s-link plane hinged mechanism (Figure 1) in the gravitational field with masses m_i $(i = 1, \ldots, s)$, controlled by a horizontal force u applied at one of the points m_j, or by a moment u applied at the j-th link. Let ξ_i, η_i be the coordinates of m_i. The problem may be to determine an action $u(t)$ which, during a given time $0 \leqslant t \leqslant T$, moves the system from the initial states $\xi_i^0, \dot{\xi}_i^0$ to a state in which all the masses m_i lie along the same straight line $\xi_i^k = (l_1 + \ldots + l_i)\, z_1$, $\dot{\xi}_i^{(k)} = 0$ inside a given interval $\alpha \leqslant z_1 \leqslant \beta$. The number T may be given; for example, $T = \theta[z] = z_2$, subject to the restriction

$$P = \int_0^{\theta} |u(\tau)|\, d\tau \leqslant 1 \qquad (2.3)$$

on the impulse P of the force $u(t)$. If in the latter case it is required to find $z\{z_1, z_2\}$ and $u(t)$ for which the problem is solvable with the smallest $z_2 = T$, we obtain an optimal speed problem.

3. Consider the observation problem. Suppose the dynamic system (2.1) is observed on the basis of the quantity

$$y = q[t, x], \qquad (3.1)$$

where y is the vector of observable coordinates y_i $(i = 1, \ldots, m)$.

The vector y is generally of lower dimension than x. Accordingly, it is assumed that x cannot be found directly from y using (3.1). The acting forces $u(t)$ are assumed known.

Problem 3.1. Given parametric equations

$$\theta = \theta[z], \quad \gamma = \gamma[x, z], \qquad (3.2)$$

which determine the required quantity γ and a time interval $t \leqslant \tau \leqslant t + \theta$. The problem reduces to determining an operator $\varphi\,[t, y(t+\tau), u(t+\tau), 0 \leqslant \tau \leqslant \theta]$, which reconstructs the quantity γ for selected z from observation of $y(\tau)$ and $u(\tau)$ in the past, i. e.,

$$\gamma\,[x(t+\theta),\ z] = \varphi\,[t,\ y(t+\tau),\ u(t+\tau);\ 0 \leqslant \tau \leqslant \theta], \tag{3.3}$$

The parameter z may be introduced with an eye to improving the quality of the observation (see below). A typical case occurs when $\gamma = x_i$, which must be known for regulation purposes. In a more general form, one may require a vector operator Γ determining the whole vector of phase coordinates x at once, i. e., $x(t+\theta) = \Gamma\,[t,\ y\,(t+\tau),\ u\,(t+\tau),\ 0 \leqslant \tau \leqslant \theta]$. It is natural to generalize the formulation of the problem to systems with infinitely many degrees of freedom. The observable vector $y(t)$ can be related to $x(t)$ in a more general manner; for example, a differential relation $dy/dt = q\,[t, y, x, u]$ taking the inertia of the measuring apparatus into account.

Example 3.1. Consider the system of Example 2.1. Suppose the rotational velocity ψ of the beam l_1 is being measured and a number $\theta > 0$ is given. The problem may be to determine operators φ_i $(i = 1, \ldots, 2s)$ which at each instant $t + \theta$ reconstruct the coordinates $\xi_i\,(t+\theta)$ and the velocities $\dot{\xi}_i\,(t+\theta)$ of all the masses m_i from the observed velocity $\dot{\psi}\,(t+\tau)$ (and force $(t+\tau)$) in the preceding time interval $t + \tau\ (0 \leqslant \tau \leqslant \theta)$, i. e., $\xi_i\,(t+\theta) = \varphi_i\,[\psi\,(t+\tau), u\,(t+\tau)]$, $\dot{\xi}_i\,(t+\theta) = \varphi_{i+s}\,[\psi\,(t+\tau), u\,(t+\tau)]$.

4. Consider the problem of 2.1 in the linear approximation. Suppose equation (2.1) is of the form

$$\frac{dx}{dt} = A(t)\,x + B(t)\,u, \tag{4.1}$$

where A, B are matrices of appropriate dimensions. Using the Cauchy formula $x\,[t] = X\,[t^0, t]\,x^0 + \int_{t_0}^{t} X[t^0,\ t]\,X^{-1}\,[t^0,\ \vartheta]\,B\,(\vartheta)\,u\,(\vartheta)\,d\vartheta$ and (2.2), we transform (4.1) into the equation

$$\int_{\bullet}^{\bullet} G\,[z,\ t^0,\ \tau]\,v(\tau)\,d_\tau\zeta\,[z,\ t^0,\ \tau] = c(z), \tag{4.2}$$

where G is expressed in terms of X and B, and the functions v and ζ are related to the required function $u(t)$. The purpose of this transformation is to interpret the left-hand side of (4.2) as the result of applying some linear operator φ, generated by the functions v and ζ, to the given element G. When this is done the given restriction should be interpreted as a restriction on the norm of φ, i. e.,

$$\rho^*\,[\varphi] \leqslant 1, \tag{4.3}$$

or

$$\rho^*\,[\varphi] = \min,\ \text{etc.} \tag{4.4}$$

Expressing (4.2) in terms of the coordinates,

$$\int_0^\theta v(\tau)\, d_\tau \zeta [z,\ t^\circ,\ \tau]\, g^{(i)} [z,\ t^\circ,\ \tau] = c_i(z), \tag{4.5}$$

the CP is conveniently interpreted as a moment problem: find a linear operator (functional) φ which satisfies conditions (4.3) or (4.4) and

$$\varphi [g^{(i)}] = c_i \ (i = 1,\ 2, \ldots) \tag{4.6}$$

Example 4.1. In example 2.1, take $m_1 = 1$, $s = 1$, $l_1 = 1$, $\theta = 1$ with the restriction (2.3) on the impulse P of the force $u(t)$; if the mass is brought to the upper equilibrium position, the linear approximation yields equations (4.5) in the form

$$\left.\begin{aligned} \varphi[\operatorname{ch}\tau] &= \int_0^\theta \operatorname{ch}\tau\, d_\tau\zeta(\tau) = c_1, \\ \varphi[\operatorname{sh}\tau] &= \int_0^\theta \operatorname{sh}\tau\, d_\tau\zeta(\tau) = c_2, \end{aligned}\right\} \tag{4.7}$$

and condition (2.3) in the form (4.3) is

$$\rho^*[\varphi] = \int_0^\theta |d_\tau\zeta(\tau)| \leqslant 1, \tag{4.8}$$

where $\zeta(\tau)$ is a function of bounded variation; $d_\tau \zeta(\tau) = u(\tau)\, d\tau$, and thus $u(t)$ has the form of instantaneous impulsive actions.

According to Hahn's theorem, the moment problem (4.3), (4.6) is solvable if and only if the inequality $|\Sigma \lambda_i c_i| \leqslant \rho\,(\Sigma \lambda_i g^{(i)})$ is satisfied for any finite set $\{\lambda_i\}$. In particular, in the finite-dimensional case, we wee that the CP is solvable for a given z when

$$\alpha[z,\ t^\circ] = \min_\lambda \rho\left(\sum \lambda_i g^{(i)}\right) > 0 \quad \text{for} \quad c(z) \cdot \lambda = 1, \tag{4.9}$$

where $\rho(g)$ is the norm in the functional space over which the operator φ is being considered.

The minimum norm $\rho^*(\varphi^0)$ is determined by the equality $\rho^*(\varphi^0) = 1/\alpha\,[z, t^0]$, and the optimum functional φ^0 itself satisfies the condition $\varphi^0\,[t^0, g^0] = \max_\varphi (\varphi\,[t^0, g^0]) = 1$, for $\rho^*(\varphi) = 1/\alpha$, where $g^0 = \Sigma \lambda_i^0 g^{(i)}$ is the solution of the problem (4.9). This condition corresponds to the maximum principle. Consequently, the CP is solvable under the restriction (4.3) if $\alpha(t^0) = \max_z \alpha\,(t^0, z) > 1$ for $z \in Z$ or if $\sup_z \alpha\,(t^0, z) > 1$ for $z \in Z$ when the upper bound is not attained. The existence of $\alpha(t^0)$ may be proved in a fairly wide class of cases.

In the finite-dimensional case the problem (4.9) may be solved numerically. In infinite-dimensional cases, which are of interest in mechanics, actual investigation of problems similar to (4.9) seems difficult although the general formulations remain in force.

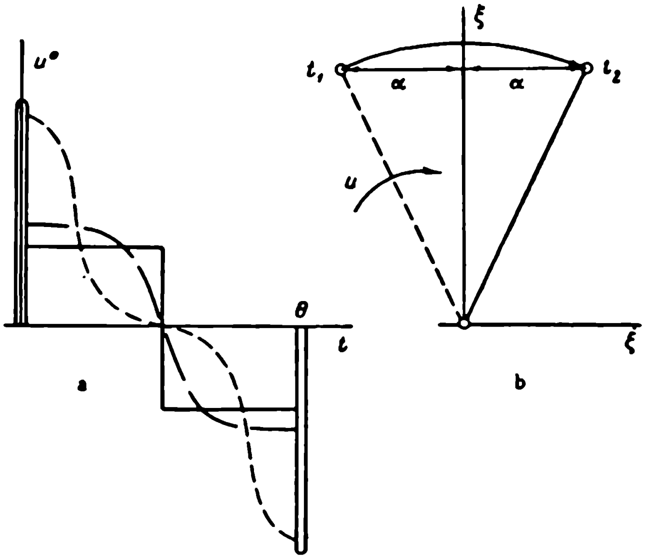
In examples (4.7), (4.8) the norm $\rho(g)$ is defined by $\rho(g) = \max_\tau (|g(\tau)|,\ 0 \leqslant \tau \leqslant \theta)$. Consequently, the smallest impulse P^0 of the force $u^0(t)$ necessary to bring the system to an equilibrium state is defined by $P^0 = 1/\alpha$, where $\alpha = \min_\lambda \max_\tau [|\lambda_1 \operatorname{ch} \tau + \lambda_2 \operatorname{sh} \tau|, \ 0 \leqslant \tau \leqslant \theta]$ for $\lambda_1 \xi^0 - \lambda_2 \dot{\xi}^0 = -1$; the problem is solvable in the time θ under condition (2.3) if $\alpha > 1$. The optimum action $u^0(t)$ is pure impulsive action $u^0(t) = \varkappa_1 \delta(t - t_1) + \varkappa_2 \delta(t - t_2)$.

Continuing the above approach, it is also very natural to study the variation of the optimum force $u^0(t)$ for different forms of the norm $\rho^*(\varphi)$. For instance, suppose the minimized quantity is estimated by the integral $\left[\int_0^\theta |u(\tau)|^p d\tau\right]^{1/p} = \min, \ (1 \leqslant p \leqslant \infty)$. For $p = \infty$ this limits the maximum action $\max_t |u(t)| = \min$; for $p = 2$ we get the minimum "power," and for $p = 1$ the minimum impulse. The behavior of the optimum action $u^0(t)$ is shown in Figure 2. In considering the problem of optimum speed from the present viewpoint, it is natural to adopt the following numerical method, by which the maximum principle actually yields initial values of the vector ψ which ensure the system's reaching the given state. By the previous reasoning, the optimum time θ^0 under restriction (4.3) is the smallest root of the equation $\alpha = 1$ (or $\alpha > 1$). We therefore take some θ and solve the problem (4.9). Using the resulting $\lambda^{(1)}$ we replace θ by an improved value $\theta^{(1)}$, so that $\alpha = 1$ is satisfied. We again solve the problem (4.9) and using the resulting $\lambda^{(2)}$ correct $\theta^{(1)}$ to $\theta^{(2)}$, and so on. The process converges under quite general assumptions. Since the quantity being minimized is a norm, problem (4.9) can be solved, for example, by the method of descent, without serious complications. This procedure may also be used to obtain the motion on the manifold (2.2) in more general cases, a game situation with respect to λ and z being typical.

5. Consider Problem 3.1 in the linear approximation. We confine ourselves to the case $u \equiv 0$, since the solution of the problem for $u \not\equiv 0$ follows from the solution of the observation problem for a free system. Suppose equations (2.1) and (3.1) have the form

$$\frac{dx}{dt} = H(t)x, \quad y = Q(t)x, \tag{5.1}$$

where H and Q are matrices of appropriate dimensions. The number θ in (3.2) is considered given and we assume $\gamma = p(z)\,x$. The problem (3.1) is now to determine a linear operator (functional) $\varphi[t, y(t + \tau)]$ such that $\varphi[t, y(t + \tau)] = p(z) \cdot x(t + \theta)$ for all t in the relevant time interval. If the conditions of the problem reduce to (4.3) or (4.4) the problem again reduces naturally to a moment problem. In fact, we have an equation $x(t + \tau) = F[t, t + \tau]\,x(t)$, $\varphi[t, Q(t + \tau)\,F[t, t + \tau]\,x(t)] = p(z)\,F[t, t + \theta]\,x(t)$. Using the linearity of φ and equating terms in the same coordinates $x_i(t)$, we again obtain the moment problem (4.6), where the elements $g^{(i)}[z, t, \tau]$ and $c_i(z, t)$ are expressed explicitly in terms of Q, F, and p. Hence follow the solvability conditions and solution methods of the OP, similar to those indicated above for the CP. This is natural, since the problems are duals of each other. The approach to these problems by way of the moment problem enables us to trace this relation conveniently and to interpret the results. For definiteness, let us consider only cases when in equations (4.1) and (5.1) u and y are scalars, B and Q are vectors b and q. The elements $g^{(i)}$ in conditions (4.6) for CP and OP can then be taken in the form $g_C^{(i)}(\tau) = \sum x_{ij}(t^0, t^0 + \theta)\, x_{jk}^{-1}(t^0, t^0 + \tau)\, b_k(t^0 + \tau)$, $g_O^{(i)}(\tau) = \sum f_{li}(t, t + \theta)\, f_{kj}^{-1}(t + \tau)\, h_k(t + \tau)$, where x_{ij}, x_{ij}^{-1}, f_{ij}, f_{ij}^{-1} are the elements of the fundamental matrices of the systems (4.1) and (5.1) respectively, and $c_c(z) = s(z) - X[t^0, t^0 + \theta]\,x^0$, $c_0(z) = p(z)$. From well-known properties of the solutions of adjoint systems it follows that when $H = -A^*$ and $b = q$ (where $*$ stands for transposition) for $t = t^0$, the elements $g_c^{(i)}$ and $g_0^{(i)}$ coincide. Consequently, the OP for $\gamma = p \cdot x$ is equivalent to the CP at the point $s(z)$ from the point x^0 where $p = s - X[t_0, t_0 + \theta]\,x^0$.



2

Conditions (4.9), which follow for the CP and OP from the L-problem of moments, can be given an intuitive interpretation, establishing a clear connection with the theory of games and problems of set separation.

6. Let us consider the OP from the following standpoint. Suppose the quantity $y(t)$ is being measured, with error $w(t)$. The quantity $r(t) = y(t) + w(t)$ will be called the total signal, $w(t)$ the noise, and $y(t)$ the effective signal. We assume that an estimate of the intensity $\rho(w)$ is given; in the sequel, for definiteness, $\rho(w) = \max_t |w(t)|$. In general, suppose the function $\eta(\tau)$ is defined for $0 \leq \tau \leq \theta$ and consider it as an element of some function space with norm $\rho(\eta)$. $\rho(\eta)$ will be called the intensity of the signal $\eta(\tau)$. Suppose the functions $\zeta(\tau)$ $(0 \leq \tau \leq \theta)$ define linear operators (functionals) on $\eta(\tau)$ with norm $\rho^*(\zeta) = \sup[\zeta[\eta],\ \rho(\eta) = 1]$. (In our case $\zeta[\eta] = \int_0^\theta \eta(\tau)\, d_\tau \zeta(\tau)$, $\rho^*(\zeta) = \int_0^\theta |d_\tau \zeta(\tau)|$.) $\rho^*(\zeta)$ will be called the intensity of the signal ζ conjugate to ρ. Conversely, if a function $\eta(\tau)$ with intensity $\rho(\eta)$ is chosen from a class of functions which define linear functionals $\eta[\xi]$ on functions $\xi(\tau)$, then the norm $\rho_*(\xi)$ will be called the initial intensity for $\rho(\eta)$. The quantity $\Delta = \sup_w (\varphi[t, w(t + \tau)]$ for $\rho(w) \leq \delta)$ will be called the error of the operator due to noise $\rho(w) \leq \delta$. The problem of observation with the least error

for $\rho(w) \leqslant \delta$ is to find an operator φ with the lowest intensity $\rho^*(\zeta)$ of the function generating φ. We shall say that the effective signal $y(t+\tau)$ $(0 \leqslant \tau \leqslant \theta)$ carries the quantity γ, if it is generated by a motion of system (5.1), where $p \cdot x(t+\theta) = \gamma$. The signal $y(t+\tau)$ carrying the quantity $\gamma = 1$ and having the least possible intensity $\rho(y)$ will be called minimum and denoted by $[y^0(t+\tau)|\rho]$. The relations indicated on page 86 are now interpreted as follows. The problem of observation of the quantity $p \cdot x(t+\theta)$ on the basis of $y(t+\tau)$ is solvable if an only if there exists $z \in Z$ such that the intensity $\rho(y^0)$ of the minimum signal $[y^0(t+\tau)|\rho]$ is nonzero for all relevant t. The lower limit Δ^0 of the error Δ for $\rho(w) \leqslant \delta$, when $p \cdot x(t+\theta)$ is being determined from the total signal $r(t+\tau)$, is determined by $\Delta^0 = \delta/\alpha$, $\alpha = \sup_z \inf_t \alpha(t, z)$. Here $\alpha(t, z)$ is the intensity $\rho(y^0)$. If $\sup_z \inf_t \alpha$ is attained, there exists an optimum operator solution φ^0 for which $\Delta = \Delta^0$. In this case the optimum signal solution $\zeta^0(t, \tau)$ has conjugate intensity $\rho^*(\zeta^0) = 1/\alpha(t)$ and it is the only signal with $\rho^*(\zeta) = 1/\alpha(t)$ such that the operator φ^0 generated by the function ζ^0 gives the largest possible value on $[y^0(t+\tau)|\rho]$.

Example 6.1. In the system of example 4.1, suppose the velocity $\dot{\xi}(t)$ is being measured on the basis of the coordinate $\xi(t)$ (for $u \equiv 0$) and $\rho(w) = \max_t |w(t)|$. Then the minimum effective signal $y^0(t+\tau)$ is a motion $\xi^0(\tau)$ of the system $(0 \leqslant \tau < 0)$ whose final velocity is 1, and the maximum deviation $\alpha = \max|\xi(\tau)|$ during the motion is the least possible (see Figure 2). The optimum reconstructive operator on any motion $\xi(\tau)$ has the form $\dot{\xi}(t+\theta) = \varkappa_1 \xi(t+\vartheta_1) + \varkappa_2 \xi(t+\vartheta_2)$, where ϑ_1 and ϑ_2 are the instants of largest deviations of the motion $\xi^0(\tau)$, $|\varkappa_1| + |\varkappa_2| = 1/\alpha$, and $\dot{\xi}^0(\theta) = 1$ on $\xi^0(\tau)$.

Taking the duality relation into account, we have the following assertion for the CP: under the restriction $\rho(u) \leqslant 1$ the CP is solvable if an only if there exists $z \in Z$ such that the initial intensity $\rho_*(y^0)$ of the minimum signal $[y^0(t^0+\tau)|\rho_*]$ $(0 \leqslant \tau \leqslant \theta[z])$ of the observed adjoint system $\gamma = p \cdot x(t+\theta)$, $y = B^*x$, $dx/dt = -A^*x$ satisfies the condition $\rho_*(y^0) > 1$, where $p = s - X[t^0, t^0 + \theta]x^0$. With this choice of z, the optimum (with respect to the intensity $\rho(u)$) control signal $u^0(t^0+\tau)$ has intensity $\rho(u^0) = 1/\alpha$, where $\alpha = \rho_*(y^0)$, and it is the only signal with intensity $\rho = 1/\alpha$ such that the operator y^0 generated by the control u^0 gives the largest possible value on the minimum signal φ^0.

In particular, in Example 5.1 the moment $u^0(t)$ with the smallest impulse which brings the system from state $\xi^0 = \dot{\xi}^0 = 0$ to state $\dot{\xi} = 0$, $\xi = \nu$ during time θ (in the linear approximation), is the reciprocal of the smallest attainable maximum deviation $|\xi(\tau)|$ $(0 \leqslant \tau \leqslant \theta)$ a motion $\xi(\tau)$ of the same system, provided that at the final instant $\nu\dot{\xi}(\theta) = 1$. The impulses of the optimum action $u^0(t)$ must be applied at the same instants t_i when the quantity $y(\tau) = \xi(\tau)$ is measured in the adjoint observation problem, i.e., at the instants when the deviation $|\xi(\tau)|$ is a maximum in the minimum signal $y^0(\tau) = \xi(\tau)$. The optimum intensity $\rho(u^0)$ of the control signal u^0 is then proportional to the magnitude of the optimum error arising in the adjoint observation problem in the presence of noise w with intensity $\rho_*(w)$.

Thus the optimum operator φ^0 generates the largest possible $\varphi^0[y^0]$ under the most unfavorable conditions for the effective signal y^0. As indicated, this property of φ^0 follows from the solution of Problems 2.1 and 3.1, based on the theory of the L-problem of moments. It is useful, however, to interpret these results from yet another point of view.

7. Let us consider the process of observation of the quantity $\gamma = p \cdot x(t)$ on the basis of $y(t)$ as a game played between the observer and the object.

Since the noise $w(t)$ of intensity $\rho(w) \leqslant \delta$ is fairly irregular, we assume that for any choice of reconstructive operator φ the most unfavorable case of w is possible, giving an error $\Delta = \varphi(w) = \delta\rho^*(\varphi)$, i.e., the error $\Delta = \beta\delta$ is determined by the given intensity $\rho(w) \leqslant \delta$ and $\rho^*(\varphi) = \beta$. Thus the quality of the effective observation may be improved only by increasing the result of applying the operator φ to the effective signal. Accordingly, we shall not make further allowance for the noise w and consider the following game.

Game 7.1. The strategies of the first player are the operators $\varphi[t, \eta(\tau)]$ with intensity $\rho^*(\varphi) \leqslant 1$, and those of the second player are the signals $y(\tau)$ (5.1), where $x(t+\theta)\cdot p(z) = 1$. The values of t, θ, and z are fixed. The utility function is $\mu[\varphi, y] = \varphi[t, y(t+\tau)]$. The aim of the first player is to attain the largest possible value of μ, while that of the second player to attain the smallest possible value of μ.

Game 7.1 may be studied, as usual, using set-separation theorems. In this case the functional φ^0 solving the problem separates the corresponding sets. The game 7.1 has a saddle point $\{\varphi^0, y^0\}$, where $\mu[f^0, y^0] = \max_\varphi \min_y \mu$, and

$$\mu[\varphi^0, y] > \mu[\varphi^0, y^0], \quad \mu[\varphi, y^0] \leqslant \mu[\varphi^0, y^0]. \tag{7.1}$$

Consequently, φ^0 and y^0 are indeed the best strategies for both players. The optimum strategy y^0 is the minimum signal $[y^0(t+\tau)|\rho]$, the optimum strategy φ^0 is, up to a factor α, the optimum operator solution, i.e., $\varphi^0 = \alpha\varphi^0[t, y(t+\tau)]$.

For all $y(t+\tau)$ of (5.1), where $p \cdot x(t+\theta) = 1$, the operator $\alpha\varphi^0$ gives exactly $p \cdot x = 1$, and no more; the latter may occur in the general case due to the properties (7.1) of the saddle point, since one of the sets being separated, i.e., the set $\{y(t+\tau)\}$, is two-dimensional.

Thus, we have an analogy with a well-known property of active strategies of a mixed finite game. If the intensity $\rho^*(\varphi)$ is defined by $\int_0^{\theta} |d\zeta| = \rho^*$, then this analogy is carried further. In fact, let Player 1 have only pure strategies $\zeta^{\pm}(\tau, \vartheta)$ of the form $d_\tau \zeta^{\pm} = \pm\delta(\tau - \vartheta)\,d\tau$, where ϑ varies over the interval $[0, \theta]$. Then the optimum strategy $\zeta^0(\tau)$ which generates the operator $\varphi^0[y] = \int_0^{\theta} y(\tau)\,d_\tau\zeta^0$ can be interpreted as a (probabilistic) mixed strategy, where the pure strategies $\zeta^{\pm}$ appear with probabilities $|d_\vartheta\zeta^0(\vartheta)|$, i.e.,

$$\varphi^0[y] = \int_0^{\theta} d_\vartheta\zeta^0(\vartheta)\left[\int y(\tau)\,d_\tau\zeta(\tau, \vartheta)\right].$$

Let us discuss the connection between the above relations and the maximum principle. Consider, for example, the CP for the system (2.1) given $\rho(u) = \max_\tau |u(\tau)| = \min$ and $r = 1$. By the maximum principle, $u^0(\tau)$ is determined by the condition $H(u^0(\tau), \psi^0(\tau)) = \max_u H(u(\tau), \psi^0(\tau)) = \max(u(\tau)\cdot\psi^0(\tau)\cdot b)$ for $|u(\tau)| \leqslant \rho(u^0)$, where $\psi^0(t)$ is the solution of the equation $d\psi/dt = -A^*\psi$. By the duality principle, the CP corresponds to the OP, to which in turn corresponds the game 7.1, where $\mu[u, y] = \int H\,dt$, $y = \psi(t)\cdot b$.

For the type of problems under consideration, the analogy with the game 7.1 therefore points to the interpretation of the conditions of the maximum principle as conditions of the maximin of the corresponding game. It is emphasized in this case that the element ψ^0 figuring in the maximum principle must satisfy the maximin condition for the game

$$\mu = \int_0^{\theta} H(u, \psi)\, dt.$$

The connection between the CP and OP and the game 7.1 stems in particular from the fact that the L-problem of moments and the problem of set separation, on which the investigation of games is usually based, have similar formulations and are studied by methods based on the extension theorem for linear operators.

8. Let us discuss the controllability (C) and observability (O) criteria expressed in terms of the coefficients of equations (4.1) and (5.1). For the systems (4.1) and (5.1) to be fully C and O, i.e., for the existence of a control $u(t)$ and an operator φ which solve the CP and OP for any $x^0, x^{(k)}$ and p, it is necessary and sufficient that for any c condition (4.9) should be satisfied. If A, B, H and Q are constant, the necessary and sufficient condition is that the ranks of the matrices $\{L^c\} = (B, AB, \ldots, A^{n-1}B))$ and $\{R^c\} = (Q^*, H^*Q^*, \ldots, H^{*n-1}Q^*)$ be n. In the case of time-dependent and differentiable coefficients, sufficient C and O conditions in the interval $t \leqslant \tau \leqslant t + \theta$ are that for at least one τ the ranks of the matrices $\{L\} = (L_1, \ldots, L_n)$ and $\{R\} = (R_1, \ldots, R_n)$, where $L_1 = B$, $L_{k+1} = dL_k/dt - AL_k$, $R_1 = Q^*$, $R_{k+1} = dR_k/dt - H^*R_k$, be n. Of importance for the CP are the properties of uniform (with respect to t^0 or t) C and O in intervals $[t, t+\theta]$ of fixed length θ, when it is required that the intensities of the control $\rho^*(u)$ and the operator solutions $\rho^*(\varphi)$ be uniformly bounded for all $t^0 > 0$ or $t > 0$ for a selected set of bounded x^0, $x^{(k)}$ and p. This is ensured if condition (4.9) is satisfied uniformly, i.e., $\alpha(t^0) > \delta > 0$ (or $\alpha(t) > \delta > 0$). In the case of constant coefficients C and O are always uniform; in the case of variable coefficients with bounded derivatives of order up to n, sufficient conditions for uniform C and O can be formulated as follows. System (4.1) is uniformly C in intervals $[t^0, t^0 + \theta]$, if for $\mu > 0$ in each such interval there exists at least one point τ for which there exists a set of numbers j_k $(k = 1, \ldots, n)$ such that

$$\sum (l[j_k, \tau] \cdot l[j_i, \tau])\lambda_i\lambda_k > \mu \sum \lambda_i^2. \tag{8.1}$$

Here $l[j, \tau]$ are the columns of the matrix $\{L\}$. System (5.1) is uniformly O in the intervals $[t, t+\theta]$, if for some $\mu > 0$ in each such interval a point exists such that

$$\sum (r[j_k, \tau] \cdot r[j_i, \tau])\lambda_i\lambda_k > \mu \sum \lambda_i^2. \tag{8.2}$$

Here $r[j, \tau]$ are the columns of $\{R\}$. Conditional C and O are also of interest for the SP. System (4.1) may be called conditionally C in the subspace N, if for any $x^0 \in N$ and $x^{(k)} \in N$ there exists a control $u(t)$ which transfers $x(t)$ from x^0 to $x^{(k)}$. System (5.1) is conditionally O in the subspace N, if the O quantity $\gamma = p \cdot x$ for $p \in N$. Systems are conditionally C and O in N if condition (4.9) is satisfied for vectors c, corresponding to $\{x^0, x^{(k)}, p\} \in N$. For this, in turn, it is sufficient that the subspace N be contained in the subspace generated by the column-vectors of the matrices $\{L^c\}$ and $\{R^c\}$, respectively.

The C property is in a certain sense a negation of the invariance property of the coordinates of a controllable system under the action $u(t)$. Consequently, the form of the invariance conditions should be the negation of the controllability criteria. From this point of view it is natural to conclude that the coordinates of system (4.1) may be invariant under u if the rank of $\{L^c\}$ is less than n, and the coordinate $c \cdot x$ is invariant under u if the vector c is orthogonal to $\{L^c\}$.

The problem now arises of classifying all the coordinates of a linear system according to the C and O criteria. This problem is solved by canonical representation of the linear system: by linear substitution of the coordinates the system can be brought into a form in which all the new coordinates fall into four classes: {C and O}, {C only}, {O only}, {neither C nor O}.

9. Let us formulate the SP. Suppose a dynamic system is described by equation (2.1). We shall distinguish two cases: 1) complete feedback information; 2) incomplete feedback information.

In the first case the problem is formulated as follows.

Problem 9.1. To find an equation

$$g[u^{(m)}, u^{(m-1)}, \ldots, u, x, t] = 0, \tag{9.1}$$

such that the motion $x = 0, u = 0$ of system (2.1, 9.1) is asymptotically Lyapunov stable.

The stability requirements may be extended to a given region of initial perturbations, including also stability in the large. The order m of the equation may be given (the value of m can be determined from the auxiliary conditions of the problem). We shall consider only the cases $m = 0$ and $m = 1$, since even in the case of larger m the problem is usually reducible to these.

In the second case the problem can be formulated as follows.

Problem 9.2. To find a delay differential equation

$$\frac{du}{dt} = U[t, y(t+\vartheta), u(t+\vartheta); -\theta \leqslant \vartheta \leqslant 0], \tag{9.2}$$

such that the motion $x = 0, u = 0$ of systems (2.1, 3.1 and 9.2) is asymptotically Lyapunov stable.

Example 9.1. The following problem can be formulated for system (2.1): suppose the angular velocity $\dot{\psi}(t)$ is being measured; the regulating moment $u(t)$ is applied to the link l_1. Find a regulation law $du/dt = U[t, \dot{\psi}(t+\vartheta), u(t+\vartheta)]$, $(-1 \leqslant \vartheta \leqslant 0)$, which ensures asymptotic stability of the system in the position $\xi_i = \dot{\xi}_i = 0$ $(i = 1, \ldots, s)$. Stable realization of the state $\xi_i = \dot{\xi}_i = 0$ using only $\dot{\psi}(t)$ and $u(t)$ is impossible, unless an aftereffect is introduced in the regulation law.

10. Consider Problem 9.1 in the linear approximation, for $m=1$. The problem is to select a regulation law

$$\frac{du}{dt} = A_1 x + B_1 u, \tag{10.1}$$

for which the motion $x = 0, u = 0$ of systems (4.1, 10.1) is asymptotically stable. In general, the SP does not have a unique solution. A convenient approach to the problem, which singles out one definite solution, consists in parallel solution of the problem of analytical construction of the regulator (10.1), which minimizes the integral

$$I = \int_0^\infty \omega\,[t,\ x(t),\ u(t),\ \dot{u}(t)]\,dt, \tag{10.2}$$

where ω is a quadratic form. This problem can be solved effectively, obtaining simultaneously a Lyapunov function $v\,(t, x, u)$ satisfying the condition $dv/dt = -\omega\,[t, x, u, \dot{u}]$ which can be used in the further development of the problem. A sufficient stabilization condition for a system with variable coefficients is the existence of a number $\theta > 0$ such that system (4.1) is uniformly controllable in each interval $[t^0, t^0 + \theta]$ $(t^0 > 0)$ (in the case $m = 1$ sufficient regularity of $B\,(t)$ must also be assumed). The stability of a closed system is then found to be uniform; this is important in passing to a nonlinear system. If (4.1) has constant coefficients, the known structure of the motions makes it possible to formulate a more precise stabilization (S) criterion, with a clear geometrical meaning. In the process of stable regulation in the present case, it is sufficient to control the intrinsically unstable coordinates of the object. The S criterion therefore reads as follows: in order that the system (4.1) be an S regulator (10.1) for $A = \mathrm{const}$ and $B = \mathrm{const}$, it is necessary and sufficient that the root subspace $K\,\{\lambda_i^{+0}\}$ of the matrix A, corresponding to its eigenvalues λ_i^{+0} with nonnegative real part, be contained in the subspace generated by the column-vectors of the matrix $\{L^c\}$. Here and below the term root subspace $K\,\{\lambda_j\}$ corresponding to a set $\{\lambda_j\}$, means the following: let T be a matrix which transforms into Jordan form $G = T^{-1}AT$. Then $K\,\{\lambda_j\}$ is the linear subspace generated by the columns T_j with the same indexes as those of λ_j in the matrix G. If A is simple in structure the subspace $K\,\{\lambda_j\}$ is generated by the corresponding eigenvectors S_j of the matrix A.

The stabilization criterion can also be given a somewhat different form. Let $\lambda_1, \ldots, \lambda_m$ be all the eigenvalues of the matrix A with $\mathrm{Re}\,\lambda_i > 0$, and suppose each eigenvalue $\lambda_i^* = -\lambda_i$ of the conjugate matrix A^* has q_i eigenvectors $S_j^{(i)}$. System (4.1) is stabilized by the control (10.1) if and only if the rank of each matrix $L_{jk}^{(i)} = S_j^{(i)}\,b_k$ is q_i, where b_k are the columns of the matrix B.

Consider Problem (9.2) in the linear approximation.

A sufficient condition for stabilization of system (4.1) by the regulator

$$\frac{du}{dt} = \int_{-\theta}^{0} \{y\,(t + \vartheta)\,d_\vartheta A_1\,(t, \vartheta) + u\,(t + \vartheta)\,d_\vartheta B_1\,(t, \vartheta)\} \tag{10.3}$$

is that in any interval $[t, t+\theta]$ this system is uniformly C and uniformly O. Here too the stability of a closed system is found to be uniform.

If A, B and Q are constants a more precise stabilizability criterion may be formulated, since it is then sufficient to control only the unstable coordinates and to observe only those coordinates necessary for such control.

Let $L[j_1, \ldots, j_m]$ $(m \leqslant r)$ denote the linear subspace generated by the column-vectors of the matrix $\{L^c\}$ with indexes $j = j_k + ir$ $(i = 0, \ldots, (n-1))$. We denote by R the subspace generated by the column-vectors of $\{R^c\}$ (for $H = A$). System (4.1, 5.1) is an S-regulator (10.3) if there exists a set of numbers j_k such that $K\{\lambda_i^{+0}\} \subset L[j_1, \ldots, j_m] \subset R$. Let us formulate another form of the criterion (for definiteness we assume that u and y are scalars). Let $\lambda_1, \ldots, \ldots, \lambda_m$ be all the eigenvalues of the matrix A with $\operatorname{Re} \lambda_j > 0$, one eigenvector $S^{(j)*}$ corresponding to the eigenvalue $\lambda_j^* = -\lambda_j$ of the conjugate matrix A^*. If all the scalar products $bS^{(j)*}$ and $qS^{(j)}$ are nonzero, the system $dx/dt = Ax + bu$, $y = qx$, is an S-regulator (10.3). The nature of the functions $A_1(\xi$ and $B_1(\vartheta)$ in (10.3) is determined by the function space $\rho(\eta)$ in which the corresponding L-problem of moments is being solved.

11. Let us discuss the CP, OP and SP for nonlinear systems. A general theory of nonlinear C, O, and S problems does not seem to exist at the present time. One may speak only of the theory of C, O, and S of quasilinear systems, obtained by investigating the problems in the linear approximation. One may then establish an analogy with the theory of stability in the first approximation and with the method of small parameters in the theory of oscillations.

Consider the systems

$$\frac{dx}{dt} = Ax + Bu + f[t, x, u], \quad y = Qx + q[t, x], \tag{11.1}$$

$$\frac{dx}{dt} = Ax + Bu + \mu f[t, x, u, \mu], \quad y = Qx + \mu q[t, x, \mu]. \tag{11.2}$$

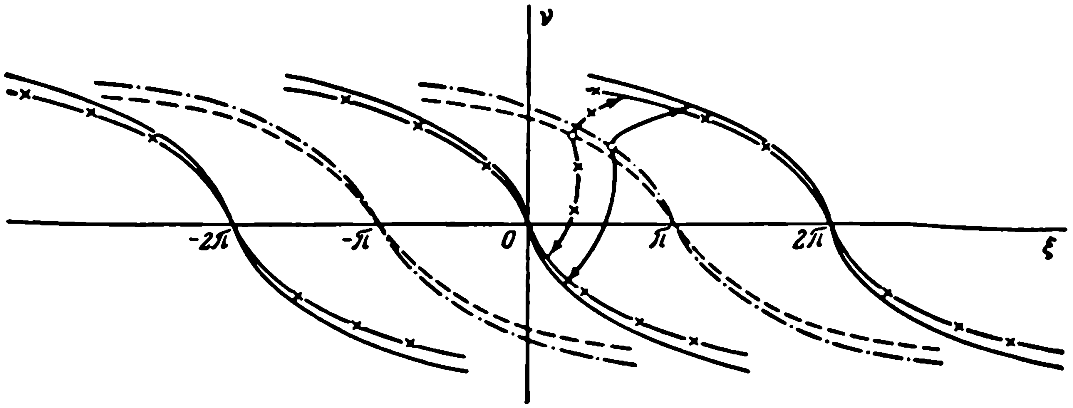
System (11.1) will be considered in the neighborhood of the point $x = 0$, $u = 0$, assuming that the order of smallness of f and q with respect to x and u is higher than one; system (11.2) will be considered in a finite, sufficiently large region, assuming μ to be a small parameter. In the event that we expand the solutions in series, the functions f and q will be assumed analytic. The basic result regarding the CP, OP, and SP for systems (11.1) and (11.2) is: if the systems are completely controllable, completely observable, and stabilizable in the linear approximation, then the quasilinear systems are also C, O, and S; the quasilinear solutions may be expanded in series (for example, for (11.2) in powers of μ). When solving the SP of (11.1) and (11.2) in the nonstationary case one should check the uniform C and O of the linear approximations.

The above cases will be called noncritical C, O, and S cases for (11.1) (11.2). As an example we may consider the problem of an unbalanced beam (Figure 3). The equation of motion has the form $d^2\xi/dt^2 = -\mu \sin \xi + u$. In the linear approximation the system is C and O. Therefore for any initial conditions $\xi^0, \eta^0 = \xi^0$, a control $u(t) = \Sigma u_k(t) \mu^k$ can be constructed which brings the beam to the state $\xi = 0, \eta = 0$.

One may constructs an operator which reconstructs the velocity $\eta(t+\theta)$ on the basis of the coordinate $\xi(t+\tau)$ $(0 \leqslant \tau \leqslant \theta)$, $\eta(t+\theta) = \Sigma \mu^k \varphi_k [t, \xi(t+\tau)$.

$u(t + \tau)]$, and finally construct a regulator $du/dt = \Sigma\mu^k f_k [t, \xi, \eta, u]$ which stabilizes the beam in the position $\xi = \eta = 0$ and minimizes, for example, the quantity $\int\limits_0^\infty [\xi^2(t) + \eta^2(t) + u^2(t) + \Sigma\mu^k\omega_k(t, \xi, \eta, u)]\,dt$.

It is interesting to note that the dependence of the solutions of the C, O, and S problems on μ may turn out to be fairly regular even if the solutions themselves are fairly irregular, for example, discontinuous. As an example, we consider the problem of the fastest relaxation of an unbalanced beam in the lower stable equilibrium position by a moment $u(t)$ of bounded magnitude, $\max_t |u(t)| \leqslant 1$. The phase plane of the optimal system has the form depicted in Figure 3, where the trajectories of the optimal system for $\mu = 0$ are shown (the solid curves represent the motions, the broken curves, the isochrones, i.e., the curves separating regions of attraction of equilibrium positions which differ from each other by a period π). The corresponding curves for small μ are also shown. The equations of these curves have the form of series in μ; the equations of the lines for $\mu = 0$ are generators.



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We mention that it is also possible to formulate a certain controllability criterion for nonlinear systems of a more general form than (11.1, 11.2). Namely, if the first approximation system at the point $x = 0$ is fully controllable for $dx/dt = f[t, x, \mu]$, and in the neighborhood of each motion $x(t)$ generated by a control $u(t)$ of a certain class this system is uniformly completely controllable, then for each point x^0 of the given region G there exists a control which brings the motion $x(t)$ to the point $x = 0$. For fairly well-behaved systems this criterion can be expressed as conditions on the partial derivatives $\partial f_i/\partial x_i$ and $\partial f_i/\partial u_k$. If it is satisfied and the existence of $u(t)$ as a function of time is affirmed, we may ask whether it is possible to synthesize a stable system with a sufficiently smooth control law $u = g[x]$.

This question is partly answered by the following results. If for the system $dx/dt = f^{(1)}(x) + f^{(2)}(x)\,u$ the problem of optimal control $\int\limits_0^\infty [g^{(1)}(x(t)) + g^{(2)}(x(t))\,u^2(t)]\,dt = \min$ has a unique solution in the region G (the problem has a solution when the above criterion is satisfied), it is possible to synthesize a system stable in G with a smooth control function. The latter results, however, are more of the nature of existence theorems, since actual construction of $u(t)$ following the proof involves serious difficulties.

12. Cases in which C, O, or S systems (11.1), (11.2) cannot be solved in the first approximation are naturally called critical.

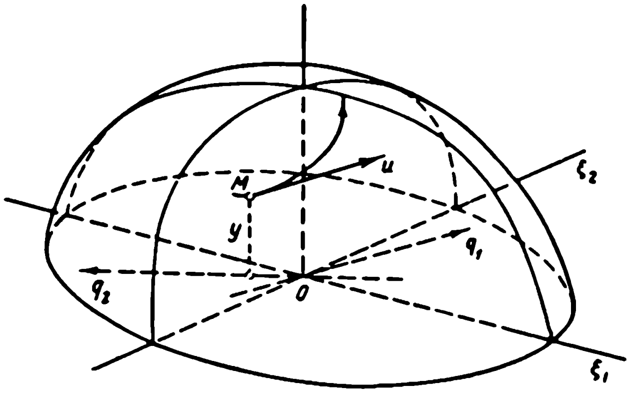
We shall only consider crticial S-cases for system (11.1) in the case of complete feedback information. The basic result can be given a clear geometrical formulation.

Consider again the matrix T which reduces A to Jordan form and the space $W\{L^i\}$ generated by the columns L_i^s. If all the vectors T_i corresponding to $\mathrm{Re}\,\lambda_i > 0$ are in W then the unperturbed motion $x = 0$ of the system $dx/dt = Ax + Bu + f\,[x, u]$ is stabilized by the regulator $u = g\,[x] = Px$ independently of f; if at least one vector T_j with $\mathrm{Re}\,\lambda_i > 0$ is not in W, stabilization is impossible; if all T_i with $\mathrm{Re}\,\lambda_j = 0$ are in W, and at least one vector T_i with $\mathrm{Re}\,\lambda_i > 0$ is not in W, then the possibility of S and the solution $u = g\,[x]$, if it exists, is determined by higher order infinitesimals of f. Thus, there is a full analogy with Lyapunov's stability criteria on the basis of the first approximation, the problem being solved here naturally by that part of the spectrum of the matrix A of the first approximation whose root vectors T_i are not contained in W. These decompositions of the space are consistent, since both W and the subspaces generated by the vectors T_i are invariant subspaces of the linear approximation system.

The investigation of critical stabilization cases is developed in analogy with Lyapunov's well-known theory. In this case, however, the possible variation of the regulator u, even within the limits of linear control $u = Px$, introduces essential new features.

13. Let us discuss the form assumed by the above relationships in the case of controllable mechanical systems. We confine ourselves to motions of holonomic systems in the neighborhood of equilibrium, assuming that only one control action is applied to the systems and that there is only one observable coordinate (u and y are scalars). Suppose the system is described by generalized coordinates $q_i\ (i = 1, \ldots, s)$, which vanish in the equilibrium position. We shall say that the system is controlled by the coordinate q_j if the term depending on u appears in the Lagrange equation corresponding to this coordinate, and that it is observed on the basis of q_j, if $y = q_j$. In the sequel it is convenient to assume that in the first approximation the kinetic energy is expressed in terms of the coordinates q_i as a sum of squares. We reduce the system in the linear approximation to normal coordinates ξ_i, $\eta_i = \dot{\xi}_i$ and obtain $\xi_i = \eta_i$, $\dot{\eta}_i = \lambda_i \xi_i + \alpha_i u$. The conditions of complete C assume the form $\lambda_i \neq \lambda_j$ and $\alpha_i \neq 0$, i.e., the system is completely C by a single coordinate q_j if and only if all its eigenvalues λ_i are distinct and the axis q_j is not orthogonal to any canonical axis ξ_i of the characteristic motions of the free system. Similarly, a system is completely O on the basis of the coordinate q_j if an only if all the numbers λ_i are distinct and the axis q_i is not orthogonal to the characteristic canonical axes ξ_i. A system is completely O on the basis of the velocity q_j, if in addition to the above conditions there are no zero eigenvalues λ_i (Figure 4). In particular, under these conditions there always exists a control in the form of a finite set of pulses $u(t) = \Sigma\, x_k \delta\,(t - t_k)$, and any coordinate q_i or velocity q_i can be reconstructed by measuring the observable coordinate $q_j = y\,(t)$ or velocity q_j at discrete instants t_k. The stabilization conditions assume the following form: 1) If $\lambda_i \neq \lambda_j$ and $\alpha_i \neq 0$, the system is stabilizable by the control $u = \Sigma\,(\beta_i q_i + \gamma_i q_i)$; 2) if $\lambda_i \neq \lambda_j, \alpha_i \neq 0$ and the axis q_k is not orthogonal to ξ_i

(or also $\lambda_i \neq 0$), the system is stabilizable by the control $du/dt = \int_{-\theta}^{0} \{q_k(t+\vartheta)\, d_\vartheta \beta(\vartheta) + u(t+\vartheta)\, d_\vartheta \gamma(\vartheta)\}$ (or $du/dt = \int_{-\theta}^{0} \{\dot{q}_k(t+\vartheta)\, d_\vartheta \beta(\theta) + u(t+\vartheta) \times d_\vartheta \gamma(\vartheta)\}$). In particular, the system in Example 2.1 is Sturmian; all the necessary conditions are satisfied, and it can therefore be stabilized by a single force u or moment u in the neighborhood of the upper unstable equilibrium position, including the case when the only source of information is measurement of one of the coordinates ξ_i or the velocities $\dot{\xi}_i$. 3) If $\lambda_i = \lambda_j$ or some $\alpha_i = 0$, then if $\lambda_i > 0$ the system is unstabilizable; if $\lambda_i \leqslant 0$, we have a critical (S) case, and approximations of higher orders must be investigated. The situation is similar in the case of C, O, or S by several coordinates.



4

Let us consider how the principle of duality between C and O is transformed here. Suppose we have a system $\dot{\xi}_i = \eta_i$, $\eta_i = \lambda_i \xi_i$. The adjoint system has the form $\dot{\xi}_i' = -\lambda_i \dot{\eta}_i'$, $\dot{\eta}_i' = -\dot{\xi}_i'$, and after renaming the coordinates and the velocities and changing the sign of the time we obtain the initial system. According to the duality principle, therefore, a system is controllable by a coordinate q_i if and only if it is observable on the basis of this coordinate.

It is of interest to study the stabilization of a mechanical system by for forces of specific types, and also to study the influence of forces of different types on the C, O, and S properties. The problem of stabilization of mechanical holonomic systems by dissipative forces has been studied in detail. From the viewpoint of general stabilization theory, the results may be interpreted and extended as follows. Suppose the system is stable for $u \equiv 0$, but not asymptotically. It is S up to asymptotic stability by forces of arbitrary nature if and only if it is S by dissipative forces. A system is S by a dissipative force on a coordinate q_i if and only if it is observable on the basis of this coordinate. S by a dissipative force is interpreted as the analytical construction of a regulator $\int_{0}^{\infty} [\omega(\dot{q}(t)) + u^2(t)]\, dt = \min$, the minimizing functional having precisely the meaning of dissipated energy.

Application of dissipative and gyroscopic forces usually improves the C, O, and S of a system. Consider for example the system of Figure 1, but assume that it is not two-dimensional, but each link has two degrees of freedom and may rotate in an arbitrary direction. Then some of the eigenvalues λ_i will naturally be equal, and the system will not be C and S by a single force ξ or O on the basis a single coordinate ξ. If, however, one introduces, say, a dissipative force in the lower link along an axis forming an acute angle with the direction of the force u or with the direction of the moment u in the horizontal plane, the system will be C, O, and S.

The system also becomes C, O, and S if a gyroscopic effect, due to a flywheel rotating about the axis l_1, is introduced.

14. Consider the stabilization of systems with aftereffect, described by equations with time delays (delay equations). The discussion of this problem is useful from two points of view. On the one hand, systems with delay are essentially systems with infinitely many degrees of freedom, and therefore involve some of the difficulties associated with such systems. At the same time, systems with aftereffects possess many regular properties by dint of which they resemble systems described by ordinary differential equations. On the other hand, when considering the OP, systems of delay equations acquire a certain new interpretation. Here we shall only discuss the SP, assuming for definiteness that the control u is a scalar. Suppose the system is described by delay differential equations

$$\frac{dx}{dt} = Ax(t) + A_1x(t-\theta) + bu.$$

An eigenvector of the system (14.1) corresponding to the eigenvalue λ_i is a nonzero vector S_i such that $x(t) = S_ie^{\lambda_it}$ satisfies equation (14.1) (when $u \equiv 0$). The eigenvector corresponding to the eigenvalue λ_i^* of the conjugate system is a nonzero vector S_i^* satisfying the equation

$$\frac{dx^*}{dt} = -(Ax^*(t) + A_1x^*(t+\theta)), \tag{14.2}$$

for $x^*(t) = S_i^*e^{\lambda_i^*t}$. When investigating the SP for system (14.1), one should separate from the system, considered in the function space $x(t+\tau)$ $(-\theta \leqslant \tau \leqslant 0)$, the finite subsystems which describe the behavior of unstable components of the motion. In these subsystems the force $u(t)$ ensures S of system (14.1). We then obtain the following S-criterion. Suppose each eigenvalue λ_i^* with $\mathrm{Re}\,\lambda_i^* \leqslant 0$ has only one eigenvector S_i^* (otherwise the system is unstabilizable by a scalar control u). System (14.1) is S if and only if none of the scalar products bS_i^* for $\mathrm{Re}\,\lambda_i^* \leqslant 0$ vanish. The S-criterion generalizes to the case of a vector control u, as well as to the case of incomplete feedback information. in forms similar to those indicated above for systems of ordinary differential equations. The first-approximation S-theory can be developed similarly. Let us discuss another problem related to systems with delay. Suppose a dynamic system is described by the equations $dx/dt = Ax + Bu$, $y = Qx$. We shall consider one possible approach to the construction of a regulator with aftereffect

$$g[u^{(m)}(t), \ldots, u(t), u(t+\vartheta), y(t+\vartheta)] = 0,$$

which stabilizes only the observable state $y(t)$ of the system. This system can be compared to the delay equation $dy/dt = p[y(t+\vartheta), u(t+\vartheta)]$, which describes the motion (variation of the signal) $y(t)$. The sufficient conditions for stabilization of this equation, derived as before, are sufficient conditions for stabilization of the motion $y(t)$.

Consideration of this problem leads to the following interpretation of delay equations. Consider the system

$$dx/dt = Ax,\ y = Qx. \tag{14.3}$$

This system can be compared to the delay equation

$$dy/dt = p[y(t+\vartheta)]\ (-\theta \leqslant \vartheta \leqslant 0), \tag{14.4}$$

which describes the observable motion $y(t)$. The nature of the right-hand side of equation (14.4) (discrete or continuous distribution of delays) is determined by how the norm of the operator solution is selected when the L-problem of moments is used for solving the OP. Conversely, every delay equation (14.4) belonging to a fairly wide class can be regarded as an equation describing the observable motion $y(t)$ of some system (14.3); in particular, of a finite-dimensional system describing the variation of the elements of the canonical expansion of the motion (14.4), which correspond to some set of eigenvalues $\{\lambda_i\}$. From this point of view it is useful to consider the indicated canonical expansion. In addition, this method provides estimates characterizing the coefficients and properties of the motions (14.3) and (14.4), as well as estimates characterizing the approximate solution of problems of analytical construction of an optimum regulator for the system (14.1), when it is approximated by finite-dimensional canonical expansion systems.

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A.M. Letov

OPTIMAL CONTROL AND STABILITY

I. FORMULATION OF THE BASIC PROBLEM

§ 1. In place of an introduction

The concept of Lyapunov stability of motion remains the most fundamental in modern science, particularly in the field of automatic control /1—8/. During the past 70 years this concept has undergone the most intense development; a particularly large contribution to stability theory after Lyapunov is due to the Soviet school of N.G. Chetaev.

The importance of stability theory was enhanced anew with the development of the theory of optimal control (TOC). This theory arouses great interest among engineers and scientists /9—27/. Its essential goal is to work out rigorous algorithms with a view to realizing the optimal operating schedules of machines, factories, economic regions, and any other purposeful man-machine combinations.

The search for optimal forms of control is now being applied to ever more complicated processes in diverse fields, whether they are determinate, stochastic or involving game situations.

Here we shall consider the simplest processes describable by ordinary differential equations.

A profound relation between the optimality problem and Lyapunov functions has been established recently for such processes /128/. Examples of this relation were indicated in /29—31, 34, 75/. Thanks to this relation, Lyapunov's stability theory, which had hitherto studied only purely topological properties of control systems, has now assumed a quantitative aspect. The latter is characterized by a functional with respect to which the system is optimized. The present paper deals with this relation.

§ 2. Brachistochronic motions

Let us consider controlled objects describable by equations of the form

$$\dot{x} = X(x, t, u). \tag{2.1}$$

Here $x\{x_1, \ldots, x_n\}$ is the state vector of the object, u is a scalar (the control function), and t the time.

The vector X of generalized forces expresses the physical laws governing the controlled object. X is assumed defined in some region

$$N(x, t, u) > 0 \tag{2.2}$$

of the phase space x, u for

$$t \in [t_0, T] \tag{2.3}$$

or

$$t \in [t_0, \infty). \tag{2.4}$$

It is appropriate to base the TOC on Lyapunov's concept of perturbed-unperturbed motion and to divide all optimal control (OC) problems into two essentially different classes /9, 27, 29, 30, 32, 34/. To the first class belong programming problems of brachistochrone motions, formulated as follows. Given equation (2.1), the region (2.2), (2.3), and boundary conditions represented symbolically by parentheses.

$$(i, f) = 0. \tag{2.5}$$

A class of comparison functions x and u is given (for example, for x, the class of piecewise-smooth and for u that of piecewise-continuous functions) on which the criterion functional (CF) is defined:

$$\Phi = \tilde{\Phi}(x. u, t)^T. \tag{2.6}$$

Functions x, u satisfying (2.1). (2.2), (2.3), (2.5) will be called admissible.

The programming problem. Among the permissible x, u, to find those which minimize the functional (2.6).

It is assumed that there exists a unique solution of the form

$$x = x^{*}[t, (i, f)], \ u = u^{*}[t, (i, f)]. \tag{2.7}$$

We call equations (2.7) brachistochronic since they are expressed in parametric form, i.e., as in the solution of Bernoulli's famous problem 267 years ago.

The programming problem is interpreted here as a mixed problem of dynamics in which the forces and motion are given partially and it is required to find the unknown forces and the motion itself.

Mathematically speaking, this is an ordinary (classical or nonclassical) variational extremum problem, and a huge and very interesting literature is devoted to its solution.

This problem is not considered in the present paper. In order to characterize it at least to some extent, we list the best known monographs; papers, of which there are more than 40,000, are not indicated.

§ 3. Basic problem of the TOC

We shall consider /33—35/ brachistochronic motion as Lyapunov unperturbed motion. This makes it possible to formulate another OC problem, essentially different from the first, which is in our opinion the basic problem of the TOC.

An obvious and easily verified fact is that brachistochrone motion (2.7) cannot be realized by a control of the form $u = u^*$, since conditions (2.5) cannot be satisfied exactly. The true motion will therefore be described by functions:

$$x = x^* + \eta, \quad u = u^* + \xi, \tag{3.1}$$

where $\eta\ \{\eta_1, \ldots, \eta\}$ is the Lyapunov perturbed motion, and $\eta(0)$, $\xi(0)$ are the initial perturbations. The equations of the perturbed motion have the form

$$\dot{\eta} = B\eta + m\xi + \Xi(\eta, \xi, t) + \varphi(t), \tag{3.2}$$

in which the right-hand sides are series expansions of the functions X.

In particular, these may be Taylor expansions, in which the matrices

$$B = \| b_{kl} \| = \left\| \frac{\partial X_k}{\partial x_j} \right\|^*,$$

$$m = \| m_k \| = \left\| \frac{\partial X_k}{\partial u} \right\|^*$$

are defined for the motions (2.7). These equations are defined in the region (2.2), (2.3).

We note that in formulating the basic problem there is no need to assume that the motion (2.7) is brachistochronic. It may be any motion agreeing with equations (2.1), which we wish to impose on the system.

For any $\eta(0) = \eta_0$, constrained (possibly) by the condition

$$\sum \eta_k^2(0) \leqslant A, \tag{3.3}$$

where A is a given number, and a given function $\xi(t)$, equations (3.2) determine what is called a transient process in automatic control theory.

One of the principal problems of automatic control theory is to control a transient process by suitable definition of ξ. It is required to find both the method and the means of cancelling a transient process in a manner which is, in some sense, optimal. In particular, it is desirable that for $t = T$:

$$\Sigma \eta_k^2(T) \leqslant \delta, \tag{3.4}$$

where δ is a specially selected number.

We therefore introduce a certain measure estimating the deviation of the true motion (3.1) from the programmed one (2.7), or a measure estimating the quality of the transient process. For this purpose we select a class G of comparison functions, on which we define the functional (3.5). From the mathematical viewpoint this class should be as wide as possible, but its size may be limited by practical considerations.

Let the functional be

$$J(\xi) = r\int_0^T W(\eta,\ \xi,\ \dot{\xi},\ t)\,dt + s\omega[\eta(T),\ \xi(T),\ T], \tag{3.5}$$

where W and ω are nonnegative and sufficiently smooth functions of their arguments, and r, s are nonnegative numbers.

The simplest problem can be formulated without allowing for the specific form and features of the current technological means on which the realization of the required solution will be based.

The basic problem. It is required to determine a differential equation

$$F[\ldots\ddot{\xi},\dot{\xi},\ \xi,\ \eta,\ t] = 0, \tag{3.6}$$

over the class G of admissible comparison functions, which expresses the control law, such that a controller constructed in accordance with the law (3.6) and attached to the object (3.2) guarantees that the functional (3.5) is a minimum for all motions starting in (3.3). This is the problem of synthesis (or analytical design) of optimal controllers.

Remark 1. Inequality (3.4) is in general not a boundary condition. The expression $\Sigma\eta_k^2(T)$ is itself a functional, and (3.4) only expresses the natural desire to bring the representative point of the system to some restricted neighborhood of its state f. In the particular case when b_{kl}, $m_k = \mathrm{const}$ or are periodic functions of t, the idealization $T = \infty$ is admissible for the functional (3.5). We may then put $\delta = 0$, and condition (3.4) means

$$\eta_k(\infty) = 0 \quad (k = 1, \ldots, n), \tag{3.7}$$

i. e., the controller (3.6) must also ensure asymptotic stability. This is concomitant with the boundedness of the functional (3.5) for $T = \infty$. If nevertheless $T < \infty$, then by setting

$$\tau = \frac{T}{T-t} - 1, \tag{3.8}$$

we reduce the basic problem to the former form. Thus, even for bounded T, we may formally speak of the stability ensured by the controller (3.6). In a sense, this will mean monotonic damping of some function $V(\eta, \xi, t)$, positive for $t \in [0, T]$.

In this case we can only guarantee the existence of a number $\delta \leqslant A$ such that inequality (3.4) is satisfied. Of course, in other respects this reduction is formal in character, and it does not essentially exclude investigation of the solutions of the problem.

Remark 2. If W does not contain derivatives of ξ, equation (3.6) degenerates and becomes algebraic:

$$F[\xi,\ \eta,\ t] = 0. \tag{3.9}$$

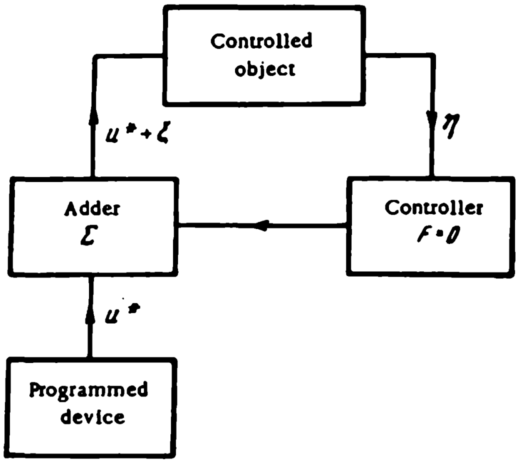
We shall say that this defines an ideal controller, i. e., one which requires a source of energy of infinite power.

By appropriate change of variables the basic problem may also be reduced to the design of an ideal controller in a number of more general cases.

Remark 3. In formulating the basic problem we have assumed that the mathematical model (3.2) has the property of "complete controllability and complete observability" /75/ or the property of "complete stabilizability" /50, 72—74/. The requirement of "complete observability" may be weakened. As regards "complete controllability" or "complete stabilizability," these seem so natural and obvious that there is no question of weakening them.

§ 4. General control scheme

Suppose that the control law (3.6) is known. The general block-diagram of a control system is given in Figure 1.



1

The brachistochronic motion is located in a programmed device. Such a device can always be built by using clocks. It emits signals which proceed to an adder. They affect programmed deviation of the control unit u^*; an additional deviation ξ is established by a controller, which utilizes information η on the true state of the controlled object. The latter consists of a suitable measuring apparatus which operate in accordance with the algorithm inherent in (3.6). The diagram represents the general idea of the current theory of automatic control.

§ 5. Selection of the criterion functional (CF)

The choice of the CF is governed by our subjective estimate as to what is desirable in the specific case in the question.

For $s=0$ the functional (3.5) is a measure of smoothness of the perturbed motions η, ξ along the brachistochrone. The presence of suitably chosen expressions in W depending on ξ, ξ is aimed at minimizing the energy consumption of the controller.

In terminal control problems the functionals (3.5) are studied for $r = 0$. They characterize the measure of deviation of the system from the terminal state f at the instant T/11, 23, 47—49, 56/.

At present, investigations are concerned mainly with quadratic functionals. From the normal point of view, a pre-selected functional should be regarded as a postulate determining the character of the resulting control theory. And as in geometry, each preassigned functional is associated with its own theory of optimal controller design. Just as at the dawn of its development mankind selected the simplest and most intuitive geometry — Euclidean geometry — for its first science of measurement, our first steps in the theory of optimal controller synthesis are based on the simple and intuitive quadratic functional.

Other important grounds for studying quadratic functionals will be discussed later. However, whatever the CF, its minimization means that the corresponding unperturbed motion will have an optimum measure of stability in the adopted sense. Consequently, the functional characterizes the quantitative aspect of the stability of the unperturbed motion.

§ 6. Time optimal control and terminal control

If we put $\delta = 0$, $rW = 1$, $s = 0$ in (3.4) and (3.5), we arrive at the time optimal control problem. For $r = 0$ we obtain the terminal control problem /11, 12, 47—49, 56/. Although formally both problems are included in the general framework of the basic problem, there are nevertheless grounds for their separate treatment in the sequel, in connection with the method of solution of the basic problem.

II. PROGRESS IN THE SOLUTION OF THE BASIC PROBLEM (BP)

§ 7. The general situation

The synthesis problem is new in comparison with the programming problem; it stems from the requirements of modern engineering and its historical development. On the basis of experience and intuition, engineers constructed, to the best of their ability, controllers of the form (3.9), and later of the form (3.6), mainly in order to maintain desirable steady-state conditions of the controlled objects. Their intuition revealed universal control principles: the principle of control by deviation (η, ξ), the principle of control by perturbation ($\varphi(t)$, and the feedback principle.

In principle, science is now occupied only with a search for improved and more accurate forms of these principles. The simplest example of improvement is the current TOC. In more complicated cases, in accordance with the cybernetic conception of N. Wiener, we are striving for perfect forms of these principles in adaptive systems. The desire for optimality is in itself not so much commendable as necessary, in view of the extremely limited material resources at mankind's disposal.

An effective solution of the synthesis problem has been or may be obtained in a small, though nevertheless impressive, number of cases. Before proceeding to their description, we briefly describe the mathematical methods used. In so doing we shall be guided by the experience reported in the literature as to the measure of success of these methods.

We sincerely apologize to the mathematicians for such a primitive approach to evaluation of these methods. For applied scientists such an approach is inevitable and it is justified by the desire to obtain, as quickly as possible, an expression of the concrete content of the problem in analytical form. Evaluation of the potential possibilities of the methods remains the task of the mathematicians themselves. The most we can indicate here amounts only to the substance, possibly, of inadequately justified guesses.

§ 8. Methods of variational calculus

The first, simplest, synthesis problems were solved by methods of variational calculus /33/. The mathematical school of L.S. Pontryagin has developed these methods considerably and given them an elegant description in the form of a single theorem, the so-called "maximum principle" /9/.

The maximum principle has brought about improved understanding and solutions of modern complicated variational applied problems defined in closed spaces, and has been used extensively to solve programming problems. An important contribution to the development of variational methods is due to V.A. Troitskii /99—104/ and V.F. Krotov /91/. Variational methods possess also a number of other important advantages over other methods. For example, they yield an unambiguous proof of the following fact. If the function W in (3.5) contains an m-th order derivative of ξ, equation (3.6) is also of order m. However, practical application of the methods involves difficulties which have not yet been overcome, mainly, the need to express the auxiliary Lagrange multipliers in terms of the phase coordinates x. These multipliers are not related to the content of the problem and have no part whatsoever in its formulation. The famous two-point boundary-value problem therefore stands in our way as a fortress attacked many times, but as yet unconquered.

§ 9. The Lyapunov-Bellman method

The method of dynamic programming (DP) or that of V.F. Krotov seems more attractive in a number of more special cases /10—16, 25, 27, 91/.

The DP method involves investigation of a certain partial differential equation for the function

$$V(\eta,\ t)=\int_{t}^{T}[rW+s\delta(t-T)\ \omega]\,dt, \tag{9.1}$$

which we shall call the criterion function. This equation is the mathematical expression of Bellman's optimum principle. The DP method based on it is applicable to the investigation of many optimum processes. Of course, this method also introduces certain difficulties. For example, in the solution of the basic problem its scope is limited to functions V which are continuous and piecewise-smooth with respect to η and ξ, and this cannot be ascertained in advance. Fortunately, however, when applied to the basic problem the DP method is closely related to the direct Lyapunov method of constructing V-functions for the stability problem. The relation is as follows. As a rule, it turns out that the optimum principle is satisfied by some set of criterion functions $V(\eta, t)$ /34/. The solution of the problem is given only by those which are Lyapunov functions for closed-loop systems. Such functions are called optimum Lyapunov functions (OLF) /29, 30, 34, 43, 58/. Controllers based on such functions are optimal and ensure stability of closed-loop systems.

The Lyapunov method therefore picks out the necessary solutions from those given by Bellman's optimum principle. It is this combination of the two methods which yields an effective solution of the synthesis problem. Moreover, Lyapunov's rigorous and fundamental concept of perturbed-unperturbed motion is essential to the very formulation of the synthesis problem. In its essence, this concept, developed 70 years ago, was a natural basis for the development of classical control theory, and is now becoming the foundation for modern control theory.

We shall formulate the basic relations of the method in the form of sufficient optimum conditions /29, 43, 58/. Let the equations of perturbed motion have the form (3.5), defined for $\|\eta\| < R$, $t \in [0, \infty)$: $R > 0$ is a given number.

By a suitable choice of equation (3.9) for ξ, we must minimize the functional (3.5) $[s = 0, T = \infty]$ over all the motions in the region $\|\eta_0\| < R_0 \leqslant R$, so that the state $\eta \equiv 0$ of the closed-loop system is asymptotically stable for all η_0 such that $\|\eta_0\| < R_0$. It may turn out that $R_0 = R = \infty$. Then, if there exist a function $V^0(\eta, t)$ and a function $\xi^0(\eta, t)$ satisfying equation (3.9) such that:

1) they are defined in the region $\|\eta\| < R$;
2) the function $V^0(\eta, t)$ is positive definite;
3) at the point $\eta \equiv 0$ the function V^0 admits an infinitesimal upper limit;
4) $\operatorname{Sup}[V^0, \|\eta\| \leqslant R_0] < \inf[V^0, \|\eta\| = R]$;
5) the derivative of $\dot V^0$ along the trajectories of the system closed by the controller (3.9) satisfies the equation

$$(\dot V^0)_{\xi^0} = -W(\eta, \xi^0, t) \tag{9.2}$$

and the optimum condition

$$(\dot V^0)_{\xi^0} + W(\eta, \xi^0, t) = \min_{\xi}[\dot V^0 + W(\eta, \xi, t)] = 0, \tag{9.3}$$

then ξ^0 is an optimal control, and we have the following equation:

$$V^0(\eta_0, 0) = \min_{\xi} \int_0^\infty W(\eta, \xi, t)\, dt. \tag{9.4}$$

If equation (3.9) has been found, the stability of the closed-loop system is ensured by conditions (1—5), since V^0 is a Lyapunov function, and the optimality of the controller follows from equation (9.3), which expresses R. Bellman's optimum principle.

For $T < \infty$, the principle formulated above is still meaningful, if the definition of the function V^0 also utilizes the boundary condition

$$V^0(\eta, T) = s\omega, \tag{9.5}$$

and "stability" is interpreted as in § 3.

§ 10. Solvable cases of the synthesis problem (analytical design) /29—63/

Let us first consider steady-state motions. We assume that:

1) there are no perturbing forces: $\varphi(t) \equiv 0$;

2) the right-hand sides of equations (3.2) and the function W may be expanded in entire series in η convergent for $\|\eta\| < \rho \leqslant R_0$;

3) the equations are linear in ξ and W is a quadratic function;

4) the domain of definition $N(x, u)$ of the problem is open.

These assumptions correspond to the relations

$$\left.\begin{aligned} \dot{\eta}_k &= \sum_\alpha b_{k\alpha}\eta_\alpha + \sum_\alpha\sum_\beta C^{(k)}_{\alpha\beta}\eta_\alpha\eta_\beta + \ldots + m_k\xi, \\ W &= \sum_\alpha\sum_\beta a_{\alpha\beta}\eta_\alpha\eta_\beta + \sum_p\sum_q\sum_r C_{pqr}\eta_p\eta_q\eta_r + \ldots + c\xi^2, \quad c > 0. \end{aligned}\right\} \tag{10.1}$$

Then the OLF is represented by a series of forms:

$$V = V^{(2)}(\eta, t) + V^{(3)}(\eta, t) + \ldots + = \sum_m V^{(m)}(\eta, t), \tag{10.2}$$

where m is the order of the form, $m = 2, 3, \ldots$. The control law is determined by the formula

$$\xi = -\frac{1}{2c}\sum m_k \frac{\partial V}{\partial \eta_k}. \tag{10.3}$$

Series (10.2) and (10.3) converge for $\|\eta\| < \rho$, where ρ is sufficiently small.

To find the forms $V^{(m)}$ we must solve systems of recurrent algebraic equations, the principal of which is a system of quadratic equations of the form

$$0 = a_{\alpha\beta} + 2\sum b_{k\alpha}A_{k\beta} - \left(\sum m_k A_{k\alpha}\right)\left(\sum m_k A_{k\beta}\right). \tag{10.4}$$

It depends only on the coefficients of the linear part of equations (10.1) and the quadratic part of the expansion of the function W, and determines the coefficients of the form $V^{(2)}$.

The coefficients of the remaining forms are determined by solving the corresponding systems of recurrent linear equations. The latter are always solvable if system (10.4) can be solved. This determines the importance of the simplest problem — that of the synthesis of linear systems whose criterion functional is quadratic.

Under the conditions of the present problem vanishing perturbing forces can be taken into account. The equation for ξ then corresponds to the principle of combined control, containing signals for both the deviation η and the perturbation φ /44—46/.

For nonsteady-state motions defined for $t \in [0, T]$, or for W depending on t, the optimal Lyapunov function (10.2) and the solution (10.3) have the same form. Now, however, the quadratic equations (10.4) become Riccati equations

$$-\dot{A}_{\alpha\beta} = a_{\alpha\beta} + 2\sum b_{k\alpha}A_{k\beta} - \left(\sum m_k A_{k\alpha}\right)\left(\sum m_k A_{k\beta}\right). \tag{10.5}$$

Their solutions determine the coefficients of the form $V^{(2)}$; the coefficients of the remaining forms satisfy recurrent systems of linear differential equations.

These systems must all be integrated, starting from a definite value of the function V for $t=T$, from T to 0 /60/. The results admit effective generalizations to more complicated dependence of the equations and the function W on ξ, and to the important case in which the vector m depends on η and t and is representable by a convergent power series in η.

These constructions raise a number of mathematical problems, specific for the theory of differential equations. An example is the following /35/.

Given the form $V^{(2)} = \sum\sum A_{kl}x_k x_l$ and the system of equations $\dot{A}_{kl} = f_{kl}(A_{11}, \ldots, A_{nn}, t)$, it is required to determine necessary and sufficient conditions on the functions f_{kl} and a region of initial values $A_{kl}(0)$, for which the form $V^{(2)}$ is positive definite. This problem was solved recently by N.P. Erugin /98/.

In conclusion we note that the Lyapunov-Bellman method and all the results described above, may be rigorously generalized to hereditary /50, 72, 74/ and stochastic processes /64—71, 76—84/, which can be synthesized in conditions of only partial observability.

If the basic problem is defined in a closed space $N(x, u) > 0$, an additional mathematical problem arises, concerned with a more detailed investigation of the properties of the function V, in particular of the continuity properties of V and $\partial V/\partial x_k$ in the whole space $N(xu) > 0$, and its uniqueness. We shall not go into this here.

Thus, the Lyapunov-Bellman method provides a unified basis for the solution of the synthesis problem for optimal controllers, i.e., the problem of technological means ensuring optimal measure of stabilizability for any unperturbed motion. From the aspect of application, the method is of interest, since it allows for all possible, previously determined, technological aspects of the synthesis problem, such as the choice of technological means for realizing the controller.

III. INVERSION OF THE SYNTHESIS PROBLEM

§ 11. Formulation of the inversion problem

It has thus been proved that if a criterion functional is given in the basic problem, optimal control is provided by an optimal Lyapunov function V^0 satisfying Bellman's optimum principle. It is of interest to examine the role of Lyapunov functions from the opposite direction: can any system for which a Lyapunov function is known be considered optimal, in some sense?

Let us delineate the inversion problem more accurately. Let us assume that in the region $N(x, u) > 0$ the differential equations

$$\dot{\eta} = f(\eta, t, p) \tag{11.1}$$

of the perturbed motion of some closed-loop system are defined, where $p\{p_1, \ldots, p_m\}$ are parameters, to be chosen in some region S, such that for each $p \in S$ the state

$$\eta^* = 0 \tag{11.2}$$

is asymptotically stable.

For a wide class of functions $f(\eta, t, p)$ there exists a nondenumerable set of Lyapunov functions by means of which asymptotic stability may be detected /1—8/. Let $V(\eta, t)$ be such a function. By virtue of (11.1) its total derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum \frac{\partial V}{\partial \eta_k} f_k = -U(\eta, t, p) \tag{11.3}$$

is a negative definite (or negative semidefinite, if it ensures asymptotic stability).

The question is, given any $p \in S$, can we interpret system (11.1) as optimal, i.e., as the result of solving the synthesis problem for some other auxiliary system with a criterion functional whose minimum is

$$V(\eta_0, 0) = \int_0^\infty U(\eta, t, p)\,dt. \tag{11.4}$$

What is the possible form of the auxiliary system and the criterion functional? As we see, the inversion problem is a typical inversion problem of variational calculus known even to Darboux /97/.

§ 12. First approach to the inversion problem

The synthesis problem involves three functions, V, W, ξ. If W is given, we have the basic problem. Given V or ξ we obtain two different forms of the inversion problem. The first is the more general.

Consider the surface $V(\eta, t) = C$ and the vector

$$\operatorname{grad} V = \sqrt{\sum \left(\frac{\partial V}{\partial \eta_k}\right)^2} \cdot n. \tag{12.1}$$

Its direction is that of the external normal n to the surface $V = C$. Now let us define a nonzero vector $m(\eta, t)\,\{m_1, \ldots, m_n\}$ by

$$(m(\eta, t) \cdot \operatorname{grad} V) = 0. \tag{12.2}$$

This vector always exists if the surface $V(\eta, t) = C$ has no singular points for any t and C.

Now let us consider the synthesis problem for the auxiliary system

$$\dot{\eta}_k = f_k(\eta, t) + m_k(\eta, t)\,\zeta, \tag{12.3}$$

$$J(\zeta) = \int_0^\infty [U(\eta, t) + \alpha(\eta, t)\,\zeta^2]\,dt, \tag{12.4}$$

where $\alpha(\eta, t)$ is positive for all η, t. By the Lyapunov-Bellman method the synthesis problem is solved by the equations:

$$\begin{gathered} -\frac{\partial \psi}{dt} = U + \alpha\zeta^2 + \sum (f_k + m_k\zeta)\,\frac{\partial \psi}{\partial \eta_k}, \\ 0 = 2\alpha\zeta + \sum m_k \frac{\partial \psi}{\partial \eta_k}. \end{gathered} \tag{12.5}$$

Here ψ denotes the function

$$\psi(\eta, t) = \int_t^\infty [U(\eta, t) + \alpha(\eta, t)\,\zeta^2]\,dt. \tag{12.6}$$

This function satisfies the partial differential equation

$$-\frac{\partial \psi}{\partial t} = U(\eta, t) + \sum_k f_k \frac{\partial \psi}{\partial \eta_k} - \frac{1}{4\alpha}\left(\sum_k m_k \frac{\partial \psi}{\partial \eta_k}\right)^2. \tag{12.7}$$

By virtue of (12.2), equation (12.7) can be satisfied by putting

$$\psi = V(\eta, t). \tag{12.8}$$

In this case the control $\zeta \equiv 0$. Hence the theorem: any system of differential equations of the form (12.7), for which asymptotic stability is ensured by some Lyapunov function $V(\eta, t)$, may be interpreted as optimal, i.e., as the result of solving the synthesis problem for some auxiliary system (12.3) whose criterion functional is (12.4), if there exists at least one nonzero vector $m(\eta, t)$ perpendicular to grad V.

Remark. The fact that the control ζ vanishes identically is unimportant, since the role of control can always be transferred to a new function defined by

$$\xi + \frac{1}{2\alpha} \sum_k m_k^* \frac{\partial V}{\partial \eta_k} = \zeta \equiv 0,$$

where m^* is an arbitrary nonzero vector not perpendicular to grad V.

Let us establish an important geometrical property of the optimal Lyapunov function $V^0(\eta, t)$.

To this end we define an operator

$$D(\psi) = [(\text{grad}\,\psi)\,m]^2 - 4\alpha\left[U(\eta, t) + \frac{\partial \psi}{\partial t} + \sum f_k \frac{\partial \psi}{\partial \eta_k}\right].$$

Any positive semidefinite functional $\psi = V^*(\eta, t)$ will be called admissible if $D(V^*) > 0$ in some region containing the point $\eta \equiv 0$. Then, if $V^*(\eta, t)$ is any admissible function, the corresponding controller equation

$$\zeta^*(\eta, t) = \frac{-(\text{grad}\,V^*)\,m \pm \sqrt{D(V^*)}}{2\alpha},$$

obtained by solving the first equation of (12.5), is admissible.

In fact, if system (12.3) is closed by the above control, the given function V^* is a Lyapunov function for it, since

$$\frac{dV^*}{dt} = -(U(\eta, t) + \alpha\zeta^{*2})$$

by virtue of the trajectory of the closed-loop system (12.3). Consequently, the control $\zeta^*(\eta, t)$ determines an admissible vector $\eta(t)$. We then have the theorem: let $V^0(\eta, t)$ be an optimum Lyapunov function; then for any admissible $V^*(\eta, t)$ we have the inequality

$$V^0(\eta, t) \leqslant V^*(\eta, t). \tag{12.9}$$

Indeed, the optimum function satisfies equation (12.7). Referring to equations (12.5), it can be interpreted formally as the condition that the surface $V^* = C$ be the envelope of the family of admissible surface $V^\circ = C$ with respect to the parameter ζ^*. Inequality (12.9) then follows from the fact that V^0 and V^* are Lyapunov functions for the corresponding closed-loop systems, V° being optimal.

Alternatively, we can construct the following auxiliary synthesis system:

$$\dot{\eta}_k = f_k(\eta, t) + \frac{m_k}{2\alpha} \sum_s m_s \frac{\partial V}{\partial \eta_s} + m_k \xi, \tag{12.10}$$

$$J(\xi) = \int_0^\infty \left[U(\eta, t) - \frac{1}{4\alpha}\left(\sum_k m_k \frac{\partial V}{\partial \eta_k}\right)^2 + \alpha(\eta, t)\,\xi^2\right] dt, \tag{12.11}$$

where m is an arbitrary nonzero vector.

It is assumed that the integrand in (12.11) is positive. This is always true if

$$U(\eta, t) > \frac{1}{4\alpha}\Big(\sum_k m_k \frac{\partial V}{\partial \eta_k}\Big)^2. \tag{12.12}$$

This inequality is bound up with the rate of decrease of the function $V(\eta, t)$ along a trajectory of the system, i.e., with the rate of approach of the representative point to the unperturbed motion. Therefore, if there exists a positive number δ such that

$$|x(t)| \leqslant |x(t_0)|\, e^{-\delta(t-t_0)}, \tag{12.13}$$

there exist estimates for the function V and $\dot{V}$, characteristic for quadratic forms /5/, and inequality (12.12) is preserved. Consequently, the integral $J(\xi)$ may still be interpreted as a positive measure of stability of the unperturbed motion. The equations of the Lyapunov-Bellman method then assume the form

$$\begin{gathered}
-\frac{\partial \psi}{\partial t} = U(\eta, t) - \frac{1}{4\alpha}\Big(\sum_k m_k \frac{\partial V}{\partial \eta_k}\Big)^2 + \alpha(\eta, t)\,\xi^2 + \\
+ \sum_k \Big[f_k + \frac{m_k}{2\alpha}\sum_s \frac{\partial V}{\partial \eta_s} m_s + m_k \xi\Big]\frac{\partial \psi}{\partial \eta_k}, \\
0 = 2\alpha\xi + \sum_k m_k \frac{\partial \psi}{\partial \eta_k},
\end{gathered} \tag{12.14}$$

where ψ denotes the function

$$\psi(\eta, t) = \int_t^\infty \Big[U(\eta, t) - \frac{1}{4\alpha}\Big(\sum_k m_k \frac{\partial V}{\partial \eta_k}\Big)^2 + \alpha(\eta, t)\,\xi^2\Big] dt. \tag{12.15}$$

The function $\psi = V$ satisfies equation (12.14); the control ξ is defined by the formula

$$\xi = -\frac{1}{2\alpha}\sum_k m_k \frac{\partial V}{\partial \eta_k}. \tag{12.16}$$

To summarize, in many cases we can construct a nondenumerable set of auxiliary systems whose synthesis yields the given system (11.1). For all these systems, however, the minimum value of the criterion functional in the synthesis problem is unique and is defined by (11.4).

§ 13. Second approach to the inversion problem

An alternative approach to the inversion problem is based on explicit selection of the control $\xi(\eta, t)$. We are given the equations of perturbed motion of the controlled object

$$\dot{\eta}_k = f_k(\eta, t, \xi) \qquad (k = 1, \ldots, n), \tag{13.1}$$

and a control law

$$\xi = \xi(\eta, t). \tag{13.2}$$

It is required to find all the functions $W(\eta, t, \xi)$ for which the integral

$$J(\xi) = \int_0^\infty W(\eta, t, \xi)\, dt \tag{13.3}$$

is a minimum.

For the case of linear and stationary equations (13.1) and (13.2) the inversion problem has been considered by Ya. Kurtsveil' /39/ and R. Kalman /96/.

It is easy to see that substitution of ξ from (13.2) in (13.1) reduces the inversion problem to the form just studied, since the closed-loop system (13.1), (13.2) is asymptotically stable.

However, this formulation of the problem may be of some practical interest in itself. The point is that the control law (13.2) is often based on various considerations of a purely practical nature. In these cases it is interesting to ascertain the possible meaning of the optimality of the closed-loop system (13.1), (13.2).

The inversion problem is easily solved in this formulation if optimality can be established on the basis of the sufficient Lyapunov-Bellman conditions.

Indeed, let us try to find a function W of the form

$$W(\eta, t, \xi) = U(\eta, t) + \varphi(\eta, t) f(\xi, t), \tag{13.4}$$

where φ, f are known positive functions.

The sufficient optimum conditions are

$$-\frac{\partial V}{\partial t} = U + \varphi f + \sum f_k \frac{dV}{\partial \eta_k}, \tag{13.5}$$

$$0 = \varphi(\eta, t) \frac{df}{d\xi} + \sum \frac{df_k}{d\xi} \frac{\partial V}{\partial \eta_k}. \tag{13.6}$$

If $\xi(\eta, t)$ (13.2) is an optimal control, the corresponding Lyapunov function is

$$V(\eta, t) = \min_{\xi} \int_t^\infty W(\eta, t, \xi(\eta, t))\, dt, \tag{13.7}$$

which satisfies the partial differential equation

$$R(\eta, t) + \overline{\sum} Y_k \frac{\partial V}{\partial \eta_k} = 0, \tag{13.8}$$

where we have denoted

$$R = \left[\varphi \frac{df}{\partial \xi}\right]_{\xi(\eta, t)}, \quad Y_k = \left[\frac{\partial f_k}{\partial \xi}\right]_{\xi(\eta, t)}. \tag{13.9}$$

Knowing even only one function (10.27) satisfying equation (13.8), equation (13.5) yields a function $W(\eta, t, \xi(\eta, t))$ which establishes the meaning of the optimality of the control (13.2). The function W must be positive semidefinite, or positive definite.

It may occur that for some choice of the functions f, φ the solution of the inversion problem is unique.

§ 14. A practical result

Suppose an asymptotically stable system (11.1) is given. Consider some positive semidefinite function $U(\eta, t)$ and define

$$\varepsilon(p) = \int_0^\infty U(\eta, t)\, dt \tag{14.1}$$

as the criterion function of the transient process (11.1). Then, using the given $U(\eta, t)$ we find the Lyapunov function from the Lyapunov equation

$$\frac{\partial V}{\partial t} + \sum f_k(\eta, t) \frac{\partial V}{\partial \eta_k} = -U(\eta, t). \tag{14.2}$$

From the foregoing

$$\varepsilon(p) = V(\eta_0, 0, p). \tag{14.3}$$

The criterion function depends essentially on η_0. We can give it a more convenient, invariant form, by considering the hypersphere $\Gamma |\eta_0| = 1$ and calculating the integral

$$\bar{\varepsilon}(p) = \int_\Gamma V(\eta_0, 0, p)\, d\eta_{10}, \ldots, d\eta_{n0}. \tag{14.4}$$

It depends only on the parameters p. Hence the possibility of minimizing $\bar{\varepsilon}(p)$ on S, or looking for $\inf \bar{\varepsilon}$ on S, as an integral quality criterion of the transient process.

§ 15. Differences of opinion with Dr. R. Kalman

In /96/ R. Kalman gives the investigation of the inversion problem preference over the solution of the basic problem (BP). In his opinion, the BP has no practical meaning, since the criterion functional is selected on an unreliable and subjective basis.

We cannot agree with this point of view. As already indicated above, there is no need to justify the choice of the criterion functional. We need only know, as in every synthesis, what properties we wish the system to have, and this is the basis of our choice. The content of the BP is not reduced merely because we do not yet have this knowledge. If the criterion functional possesses the required properties (and other properties are immaterial), we can find the optimal control law, which is unique in most cases of practical interest.

In the inverse problem we have no alternative choice other than the natural effect of the system (11.1), whose quality is expressed by the given number (11.4). Kalman's point of view therefore implies intentional rejection of possibilities which we can always effectively use if we are capable of distinguishing good from bad or better from good.

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V.M. Matrosov

DEVELOPMENT OF THE METHOD OF LYAPUNOV FUNCTIONS IN STABILITY THEORY

The object of this report is to extend Lyapunov's second method /1/ by simultaneous employment of several functions V, and of higher order derivatives of V, in the criteria for stability of motion. When this is done, the demands imposed on each function V are less stringent than those figuring in the classical Lyapunov theorem or in its generalization, which involve single function V and its first derivative. This facilitates the search for the required functions V and makes the method more flexible.

The majority of existing criteria originated in applied problems of stability of mechanical systems. Applications, however, are beyond the scope of this paper. Neither do we deal with other aspects of the development of the method of Lyapunov functions in the theory of stability of motion (or with the qualitative theory of differential equations). In particular, we do not touch upon the well-known generalizations of the theorems of Lyapunov's second method, in which the function V is determined in "piecewise" fashion in separate regions covering the neighborhood of the unperturbed motion (which are in general also interpretable as stability theorems involving several functions V). Neither do we discuss extensions of the results to systems describable by equations other than the customary ones.

The selection of the material was largely determined by the interests of the author and has no claims to completeness.

Introduction

§ 1. Given a system of equations of perturbed motion in vector form

$$\frac{dx}{dt} = X(x, t). \tag{1.1}$$

Here $x = \{x_1, \ldots, x_n\}$ is a vector in Euclidean n-space E^n with norm $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$; $X(x, t) = \{X_1(x, t), \ldots, X_n(x, t)\}$ is a function vector; $X(0, t) \equiv 0$, i.e., system (1.1) admits the unperturbed motion $x = 0$; the real functions $X_i(x, t)$ are defined and continuous in the region Γ: $|x| < H, t > 0$: ($H = \text{const} > 0$ or $H = \infty$) and satisfy the conditions of uniqueness, nonlocal continuability, continuous dependence on the initial conditions of the solutions of the system in the region Γ. Thus the perturbed motions are described by a function vector $x(t, x_0, t_0)$, which is defined and continuous for $\{x_0, t_0\} \in \Gamma$, $t \in [t_0, \infty)$ as long as $\|x\| < H$*, and continuously differentiable with respect to t.

Consider real functions $V_1(x,t), \ldots, V_k(x, t)$ defined and continuous in the region Γ together with their first derivatives with respect to t, the latter satisfying the equations of perturbed motion

$$\dot{V}_s(x, t) = \frac{\partial V_s(x, t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V_s(x, t)}{\partial x_i} X_i(x, t) \quad (s = 1, \ldots, k),$$

and vanishing in the unperturbed motion, i.e., $\dot{V}_s(0, t) \equiv 0, V_s(0, t) \equiv 0$. For these functions $V(x, t) = \{V_1(x, t), \ldots, V_k(x, t)\}$ we introduce a norm $\|V\| = |V_1| + \ldots + |V_k|$.

If the function $V_s(x, t)$ has partial derivatives up to the second order with respect to $x_1, \ldots, x_n, t$ which are continuous in Γ, and the functions $X_i(x, t)$ are continuously differentiable, we can introduce the second derivative of each function V_s with respect to t, in accordance with the system (1.1):

$$\ddot{V}_s(x, t) = \frac{\partial \dot{V}_s(x, t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \dot{V}_s(x, t)}{\partial x_i} X_i(x, t).$$

Higher-order derivatives are defined analogously. The derivatives of the functions V_s will always be assumed continuous in Γ by virtue of the equations of perturbed motion.

We shall also consider real functions $V^*(x), V'(x)$, defined and continuous in the region (H): $\|x\| < H$ of the space E^n, $V^*(0) = 0$, $V'(0) = 0$.

Let us denote the set of points $x \in (H)$ for which $V^*(x) = 0$ (or $V^*(x) > 0$), by $E(V^* = 0)$(or $E(V^* > 0)$). The set of points $\{x, t\} \in \Gamma$ for which $V(x, t) = 0$ (or $V(x, t) > 0$), will be denoted by $B(V = 0)$ (or $B(V > 0)$).

Instability

§ 2. After the fundamental work of A.M. Lyapunov (1892) /1/, the first important step in the development of the method of Lyapunov functions in the theory of stability of motion was made by N.G. Chetaev /2/. It was his idea to set up stability criteria with two functions V and use the second derivative of the function V.

The second derivative of the Lyapunov function was already used in 1902 by Duhem to obtain a criterion for equilibrium instability of dissipative systems /3/. In the course of his investigations in /3/, Duhem in effect proved the following theorem for autonomous systems

$$dx/dt = X(x). \tag{2.1}$$

Duhem's Theorem. If there exists a positive semidefinite function $V(x)$, whose first derivative $\dot{V}$ together with V may assume positive values in any neighborhood of the unperturbed motion $x = 0$, and the second derivative $\ddot{V}$ is positive semidefinite, then the unperturbed motion is unstable.

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* In the case $H = \infty$, the solutions $x(t, x_0, t_0)$ are defined and continuous for all $t \in [t_0, \infty)$.

In the paper "On the Inversion of Lagrange's Theorem" (1930) /21/, a far more general theorem was proved.

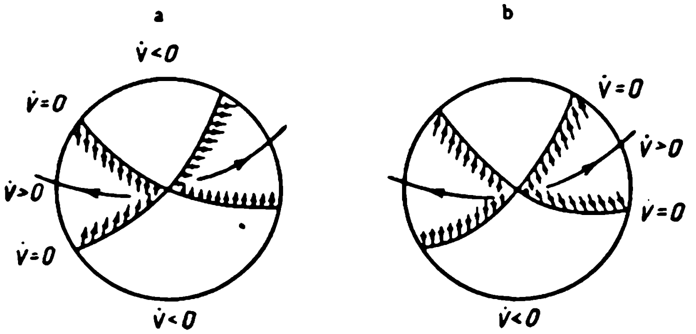
Chetaev's First Theorem. Suppose the function $V(x)$ for system (2.1) possesses the following properties.

1. In the region $E(\dot V > 0)$ defined by the inequality $\dot V > 0$, the function V is also positive.

2. Near the boundary surface $\dot V = 0$ of the region $E(\dot V > 0)$, the second derivative $\ddot V$ has a fixed sign, and has the same sign or vanishes on this surface.

Then the unperturbed motion $x = 0$ of the system (2.1) is unstable.

It is clear that the meaning of the demands imposed here on the second derivative $\ddot V$ is that they ensure the existence of perturbed motion with arbitrarily small initial perturbations, which cannot leave the region $E(\dot V > 0)$ without moving away from the unperturbed motion to a given distance. This is illustrated in Figure 1, in which near the surface $\dot V = 0$: a) $\ddot V > 0$; b) $\ddot V < 0$. For the general case, when the function V also depends explicitly on time, K.P. Persidskii introduced the concepts of sector and semisector /4, 5/ to characterize such a situation.



1

Let us consider a subregion Δ of the region $\|x\| < h,\ t > 0\ (0 < h = \text{const} < H)$ such that its intersection $\Delta(\tau)$ with any hyperplane $t = \tau$ $(\tau > 0)$ is an n-dimensional domain (open and connected), its closure $[\Delta(\tau)]$ containing the point $x = 0$ and at least one point with $\|x\| = h$.

Persidskii's definition. A region Δ is called a sector (semisector), if for any positive number ε $(\varepsilon < h)$ there exists an interior (interior boundary) point $\{x_0, t_0\}$ $(0 < \|x_0\| \leqslant \varepsilon)$ such that the integral line $x(t, x_0, t_0)$ remains inside (inside or on the boundary of) the region Δ for all $t > t_0$ for which $\|x(t, x_0, t_0)\| < h$. Let us try to characterize case a) of the figure.

Definition 2.1. The region Δ is called an absolute sector (resp. absolute semisector), if for any interior point $\{x_0, t_0\} \in \Delta$ (resp. interior or boundary point $\{x_0, t_0\} \in [\Delta]$) the integral line $x(t, x_0, t_0)$ remains inside (resp. inside or on the boundary of) the region Δ for all $t > t_0$ for which $\|x(t, x_0, t_0)\| < h$.

It is clear that the existence of a sector or semisector can be ensured by different methods, in particular, by the introduction of an auxiliary function W with suitable properties.

Modifying his first theorem in this way in the papers /2/ "A Theorem on Instability" (1934), "On an Idea of Poincaré" 1934, "On the Instability of Equilibrium, when the Force Function is not a Maximum" (1938), Chetaev published his famous two-function instability criterion (for nonautonomous systems (1.1)).

Chetaev's Second Theorem. If the differential equations of perturbed motion are such that:

1) for some function V admitting an infinite small upper limit there exists a region where $V\cdot\dot{V}>0$;

2) for some x of arbitrarily small absolute value, there exists a subregion of the region $(V\cdot V>0)$ in which a certain function $W>0$, on whose boundary $W=0$, and the total time derivative $\dot{W}$ has a definite sign; then the unperturbed motion is unstable.

One feature is worth mentioning. Chetaev considers here the infinite "closed interval of time variation," completing it by a singular point $+\infty$, and by the existence of some region in (H) means its nonemptiness for any t of the extended numerical half-line.

Problems concerned with the instability criterion were subjected to a further careful investigation by Persidskii in his dissertation /4, 5/.

Following Persidskii, we call the set of all the boundary points of the region Δ which do not belong to the cylinder $\|x\|=h$ the lateral surface ζ of the region Δ.

Persidskii's Sector Theorem. If the integral lines intersect the lateral surface ζ of the region Δ in one (in strictly one) direction, then Δ is a semisector (sector).

Corollary. Suppose there exists a function $W(x, t)$ which is positive in the region Δ and vanishes on its lateral surface ζ.

(I) If $\dot{W}(x, t)<0$ for $\{x, t\}\in\zeta,\ x\neq 0$, then Δ is a sector.

(II) If $\dot{W}(x, t)>0$ for $\{x, t\}\in\zeta,\ x\neq 0$, then Δ is an absolute sector.

If the derivative $\dot{W}(x, t)$ is such that each point $\{x, t\}\in\zeta,\ x\neq 0$ has a neighborhood U such that:

(III) $\dot{W}(x, t)<0$ for $\{x, t\}\in U\cap\Delta$, then Δ is a semisector;

(IV) $\dot{W}(x, t)>0$ for $\{x, t\}\in U\cap\Delta$, then Δ is an absolute semisector.

Persidskii's sector theorem was somewhat generalized by A. D. Myshkis /6/. He also established that the set of initial points of trajectories which cannot leave a sector or semisector through its lateral surface (with increasing time), supplemented by the origin, contains a continuum which connects the origin with the surface of the sphere $\|x\|=\rho$ of any radius $0<\rho<h$.

The concept of the sector yields the following theorem /4, 5/.

Persidskii's Instability Theorem. Suppose there exist a sector Δ and a function $V(x. t)$ with the following properties in Δ:

1) $V(x, t)$ is bounded;

2) $V(x, t)>0$ (for $x\neq 0$);

3) $\dot{V}(x, t)>\eta(V(x, t), t)$,

where $\eta(V, t)$ is a nonnegative nondecreasing function of V, for which

$$\int_0^\infty \eta(\alpha, t)\,dt=\infty \quad \text{for any} \quad \alpha>0.$$

Then the unperturbed motion $x = 0$ of system (1.1) is unstable.

Here one may avoid the requirement that the derivative $\dot{V}$ be semidefinite, by replacing condition (3) by

$$3^{*})\quad \dot{V}(x, t) > \theta(t) V(x, t), \quad \overline{\lim_{\tau\to\infty}} \int_0^{\tau} \theta(t)\, dt = \infty.$$

By making the requirement for the existence of a sector and conditions (3) and (3*) more specific, one can obtain many instability criteria convenient for applications. In particular, Persidskii's criterion generalizes the classical theorems of Lyapunov, Chetaev's well-known one-function theorem /2/, and many criteria obtained by Alma Ata mathematicians /10, 11/. Other instability criteria result from theorems of N.P. Erugin /7/, Masséra /8/, and V.I. Zubov /9/.

Let us now consider another possible choice of the second function V in the instability criterion.

For time-periodic functions $X(x, t)$ in system (1.1), N.N. Krasovskii obtained /12/ the following generalization of Lyapunov's theorem.

Krasovskii's Theorem. Suppose there exists a function $V(x, t)$ having the following properties in Γ:

1) $V(x, t)$ is periodic in t;

2) for any $t_0 > 0$ there exist points x_0 in an arbitrary neighborhood of the unperturbed motion $x = 0$ for which $V(x_0, t_0) > 0$;

3) $\dot{V}(x, t) > 0$;

4) $\dot{V}$ vanishes, if at all, only at points of a set $M \subset (H)$ which does not contain complete semitrajectories $x(t, x_0, t_0)$, $t \in [t_0, \infty)$ other than $x = 0$.

Then the unperturbed motion is unstable. In this case, if the set M is a surface defined by an equation $F(x, t) = 0$, then condition (4) is satisfied if $\dot{F}(x, t) \neq 0$ on this surface for $x \neq 0$.

Here $F(x, t)$ may obviously be regarded as an auxiliary function V which "eliminates" trajectories from the "dangerous" set M.

§ 3. In this section the functions $X(x, t)$ in (1.1) are assumed bounded in Γ.

It can be easily shown by an example that Krasovskii's theorem in the given form cannot be extended to the general case of nonstationary motions, when the right-hand sides of the equations of perturbed motion are nonperiodic time functions. Nevertheless, the author has shown /13/ that appropriate modifications may be made in Krasovskii's theorem.

We introduce the corresponding definitions of /13/.

For any positive numbers α, A $(0 < \alpha < A < H)$ we define the subregion A_α: $\alpha < \|x\| < A$ of (H).

Definition (3.1). $\dot{W}(x, t) \neq 0$ strictly (definitely) in the set $E(V^*(x) = 0)$, if in any region A_α there exists a neighborhood of the set $A_\alpha \cap E(V^* = 0)$ and a continuous function $\xi(t)$ (a positive number ξ), such that for any $t \in [0, \infty)$

$$\xi(t) > 0, \quad \int_t^{\infty} \xi(\tau)\, d\tau = \infty, \tag{3.1}$$

and $|\dot{W}(x, t)| > \xi$ in the neighborhood.

Definition 3.2. A continuous nonnegative function $\varphi(t)$ is called positive in the mean, if for any infinite system s of closed nonoverlapping intervals of the half-line $[0, \infty)$ of equal positive length we have

$$\int_s \varphi(t) = \infty.$$

The conditions of Krasovskii's theorem can be modified as follows:

1) omit the requirement of infinitely small upper limit;

2) retain completeness;

3) add: $\dot{V}(x, t) > \varphi(t) V^*(x)$, where $V^*(x) > 0$ and $\varphi(t)$ is positive in the mean (in particular, if $\varphi(t) = \mathrm{const} > 0$, this means that the position of the set M in the phase space is fixed);

4) suppose there exists an auxiliary bounded function $W(x, t)$ for which $\dot{W}(x, t) \neq 0$ strictly in the set $E(V^* = 0)$.

Then the unperturbed motion is unstable /13/.

Here condition 3) can also be maintained in the form $\dot{V} > 0$, if in addition we require that $\dot{V}(x, t)$ and $\ddot{V}(x, t)$ be bounded, and in condition 4) that the derivative $\dot{W}(x, t)$ be strictly nonzero in the set $B(\dot{V} = 0)$ in the following sense.

Definition 3.3. $\dot{W}(x, t) \neq 0$ strictly (resp. definitely) in the set $B(\dot{V} = 0)$, if for any region A_a there exist a positive number l and a continuous function $\xi(t)$ satisfying conditions (3.1) (resp. a positive number ξ) such that $|\dot{W}(x, t)| > \xi$ for $x \in A_a$, $\{x, t\} \in B(|\dot{V}(x, t)| < l)$.

Let us now combine the two directions of investigation described above.

Theorem 3.1. Suppose there exist a sector Δ and functions $V_1(x, t)$ and $V_2(x, t)$ with the following properties in $[\Delta]$:

1) $V_1(x, t)$ admits an infinitely small upper limit;

2) $V_1(x, t) > 0$ for $\{x, t\} \in \Delta$, $x \neq 0$;

3) $\dot{V}_1(x, t) > \varphi(t) V'(x)$, where $V'(x) > 0$ for $x \in (H)$ and $\varphi(t)$ is positive in the mean;

4) $V_2(x, t)$ is bounded and $\dot{V}_2(x, t) \neq 0$ strictly in the set $E(V' = 0) \cap [\bigcup_{\tau \in [0,\infty)} \Delta(\tau)]$.

Then the unperturbed motion $x = 0$ of the system (1.1) is unstable.

Here too condition 3) can be maintained in the form $\dot{V}_1(x, t) > 0$ (for $\{x, t\} \in [\Delta]$), by requiring that $\dot{V}_1(x, t)$ and $\dot{V}_2(x, t)$ be bounded and replacing condition 4) by:

4*) $V_2(x, t)$ is bounded and $\dot{V}_2(x, t) \neq 0$ strictly in the set $B(\dot{V}_1 = 0) \cap [\Delta]$.

As already mentioned, if $V_3(x, t) > 0$ in some region Δ, $\dot{V}_3 = 0$ on its lateral surface ζ, and for $\{x, t\} \in B(V_3 = 0) \cap B(0 < \|x\| < h)$ the derivative $\dot{V}_3(x, t)$ satisfies one of the conditions:

$$\text{(I)} \quad \dot{V}_3(x, t) < 0, \quad \text{(II)} \quad \dot{V}_3(x, t) > 0,$$

then Δ is a sector.

Thus, as a consequence of Theorem 3.1 we obtain a three-function instability criterion which generalizes, on the one hand, Chetaev's second theorem, and on the other, Krasovskii's theorem.

Setting $\dot{V}_1 = V_2 = V_3$, we obtain a corollary which is a modification of Chetaev's first theorem:

Corollary 3.1. Suppose there exists a function $V(x, t)$ with the following properties in Γ:

1) $V(x, t)$ admits an infinitely small upper limit, and $\dot{V}(x, t)$ and $\ddot{V}(x, t)$ are bounded;

2) there exists a region Δ in which $\dot V(x, t) > 0$, and $\dot V(x, t) = 0$ on its lateral surface ζ:

3) $V(x, t) > 0$ for $\{x, t\} \in \Delta$;

4) $\ddot V(x, t) \neq 0$ strictly in the set $B(\dot V = 0) = \zeta$; when for $\{x, t\} \in \zeta$, $x \neq 0$ one of the following conditions is satisfied:

$$\text{(I)}\quad \ddot V(x, t) < 0, \quad \text{(II)}\quad \ddot V(x, t) > 0.$$

Then the unperturbed motion $x = 0$ of system (1.1) is unstable.

§ 4. A completely different approach to the whole problem is possible. The conditions imposed on the derivatives of the function V in the Lyapunov's second method are simple differential inequalities. Thus, in the classical Lyapunov instability theorems they have the form $\dot V > \theta(V)$ ($\theta(V) > 0$ for $V > 0$ /5/, and in Persidskii's theorem the form of inequality 3), 3*).

It is natural to try to set up an instability criterion using more complicated differential inequalities and systems of differential inequalities. Though such criteria necessarily contain requirements on the functions V which are more complicated to verify, they may turn out to be more general; this enables us to use V-functions which do not satisfy the requirements of the well-known instability theorems.

The proofs may be based on Chaplygin's theory of differential inequalities /15/. Conditions for the solvability of Chaplygin's problem have now been obtained for systems of differential inequalities. Thus, in the well-known theorem of Wazewski these conditions represent monotony and continuity of the right-hand sides of the differential equations (and the corresponding inequalities) /16/.

An instability criterion with several functions V, whose derivatives $\dot V$ satisfy certain differential inequalities, was proposed by the author in /17/ and generalized in /18/. Here we formulate a criterion of this kind, in which the concept of sector is again involved.

Consider real functions $f_1(V, x, t), \ldots, f_k(V, x, t)$, defined and continuous in the region Ω: $\|V\| < \infty$, $\|x\| < H$, $t > 0$ ($V = \{V_1, \ldots, V_k\}$). In addition $f_s(0, 0, t) = 0$ ($s = 1, \ldots, k$).

Let $\xi(t) = \{\xi_1(t), \ldots, \xi_n(t)\}$ be a set of real functions which are continuously differentiable for $t \in [0, \infty)$. We set up an auxiliary system of equations

$$\frac{dy_s}{dt} = f_s(y, \xi(t), t) \quad (s = 1, \ldots, k), \tag{4.1}$$

where $y = \{y_1, \ldots, y_k\}$.

Definition 4.1. The system (4.1) possesses the $+y_1$-quasi-nonboundedness property for the functions $V(x, t)$ and a region Δ if for any positive numbers $\varepsilon, t_0, \alpha, A$ ($0 < \alpha < A < h$) and any continuously differentiable function $\xi(t)$ such that $\{\xi(t_0), t_0\} \in \Delta$, $0 < \|\xi(t_0)\| \leqslant \alpha$, $\|\xi(t)\| < A$ for $t \in [t_0, \infty)$, there exists a positive number T such that any solution $y(t)$ of system (4.1) with initial conditions $y(t_0) = V(\xi(t_0), t_0)$, defined for $t \in [t_0, t_0 + T]$, also satisfies the condition $y_1(t_0 + T) > \varepsilon$.

Theorem 4.1. Suppose there exist a sector Δ and functions $V_1(x, t), \ldots, V_k(x, t)$ with the following properties:

1) $V_1(x, t)$ is bounded in Δ;

2) $\dot V_s(x, t) > f_s(V(x, t), x, t)$ ($s = 1, \ldots, k$) for $\{x, t\} \in \Delta$;

3) each of the functions $f_s(V, x, t)$ (defined and continuous in Ω) is nondecreasing in $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$ /18/;

4) the auxiliary system $dy_s/dt = f_s(y, \xi(t), t)$ $(s = 1, \ldots, k)$ has the $+y_1$-quasi-nonboundedness property for the functions $V_1, \ldots, V_k$ and the sector Δ.

Then the unperturbed motion $x = 0$ of system (1.1) is unstable.

When we have exact equalities $\dot{V}_s = f_s(V, x, t)$ for $\{x, t\} \in \Delta$ in 2), condition 3) may be omitted.

For $k = 1$ and f not depending explicitly on x, we obtain:

Corollary 4.1. If there exist a sector Δ and a function $V(x, t)$ bounded and positive in Δ whose derivative $\dot{V}(x, t) \geqslant f(V(x, t), t)$ for $\{x, t\} \in \Delta$, where $f(V, t)$ is continuous and such that any solution of the equation $dy/dt = f(y, t)$ $(f(0, t) = 0)$ with the initial condition $y_0 > 0$, $t_0 \geqslant 0$ increases without limit with increasing time, the unperturbed motion $x = 0$ of the system (1.1) is unstable.

This proposition generalizes both variants of Persidskii's theorem.

Corollary 4.2. Suppose there exist a sector Δ and a function $V(x, t)$ bounded in Δ, whose second derivative $\ddot{V}(x, t) \geqslant aV(x, t) + 2b\dot{V}(x, t)$ for $\{x, t\} \in \Delta$, where $a, b = \text{const}$, $a \geqslant 0$, and $b \geqslant 0$ if $a = 0$, and $\dot{V}(x, t) + V(x, t) \times (\sqrt{b^2 + a} - b) > 0$ for $\{x, t\} \in \Delta$. Then the unperturbed motion $x = 0$ of the system (1.1) is unstable.

In particular, if V and $\dot{V}$ may assume positive values simultaneously for any t in an arbitrarily small neighborhood of the unperturbed motion, and $\ddot{V} \geqslant 0$ for $\{x, t\} \in B(V > 0, \dot{V} > 0)$, $B(t) \cap E(\|x\| = h) \neq \phi$, then B is an absolute sector. Taking V bounded in B and $a = 0, b = 0$, we obtain the instability criterion of /17/.

This criterion generalizes Duhem's theorem and its modification proved by Kh. I. Ibrashev /19/.

Thus, by specifying the conditions of Theorem 4.1 in various ways, different constructive instability criteria can be obtained.

Stability

§ 5. Lyapunov suggested using some "part" of the neighborhood of the unperturbed motion to investigate conditional stability.

If an absolute sector Δ is available, generalizations of Lyapunov's stability and asymptotic stability theorems to the case of conditional stability for $\{x_0, t_0\} \in \Delta$ are obtained trivially by requiring that all the conditions imposed on the function V be satisfied only in $[\Delta]$. Obviously, the absolute sector can then be defined by an auxiliary function W. In this way two-function conditional stability theorems are obtained /20/.

It is more interesting to determine the class of systems (1.1) for which the existence of a function $V(x, t)$ possessing the relevant properties in the absolute sector Δ provides information about the behavior of the perturbed motions in a complete neighborhood of the unperturbed motion. Such a class of systems has been determined by P.A. Kuz'min, in the form of linear systems with continuous and bounded coefficients /20/.

The proof is based on the fact that any solution of a linear system is a linear combination of n solutions of a fundamental system, which can be taken from Δ. The existence of an absolute sector is ensured by introducing an auxiliary function W, and so Kuz'min's theorems are two-function stability criteria.

The auxiliary function W may be incorporated into the asymptotic stability criterion in another way.

In 1952 E.A. Barbashin and N.N. Krasovskii showed /21/ for autonomous systems (2.1) that, under the conditions of Lyapunov's asymptotic stability theorem, the derivative $\dot{V}(x)$ need be only positive semidefinite, not necessarily positive definite, if the derivative $\dot{V}(x)$ may vanish only in a set M which contains no integral semi-trajectories other than $x = 0$.

An analogous theorem was published later by A.P. Tuzov /22/ and generalized by La Salle /23/. Krasovskii has extended the theorem to periodic systems /12/.

Here, as in the case of instability, the set M does not contain integral trajectories if it is a surface $F(x, t) = 0$ on which $\dot{F}(x, t) \neq 0$ for $x \neq 0$; F too can be regarded as an auxiliary V-function.

§ 6. In this section the functions $X(x, t)$ in (1.1) are assumed bounded in Γ.

As before, examples show that the theorem of Barbashin and Krasovskii in its given form cannot be extended to nonperiodic systems. The author has modified it to obtain two-function asymptotic stability criteria /13/.

Theorem 6.1. Suppose there exist functions $V_1(x, t)$ and $V_2(x, t)$ having the following properties in Γ;

1) the function $V_1(x, t)$ is positive definite and admits an infinitely small upper limit,

2) the derivative $\dot{V}_1(x, t) \leqslant \varphi(t) V^*(x)$, where $V^*(x) \leqslant 0$ and the function $\varphi(t)$ is positive in the mean (resp. a positive number);

3) the function $V_2(x, t)$ is bounded;

4) $\dot{V}_2(x, t) \neq 0$ strictly (resp. definitely) in the set $E(V^* = 0)$.

Then the unperturbed motion $x = 0$ of the system (1.1) is uniformly asymptotically stable with respect to x_0 (resp. with respect to x_0, t_0).

Condition (2) can be maintained in the form $\dot{V}_2(x, t) \leqslant 0$ by requiring the boundedness of the derivatives $\dot{V}_1(x, t)$ and $\ddot{V}_1(x, t)$ and replacing condition (4) by (4*): $\dot{V}_2(x, t) \neq 0$ strictly (resp. definitely) in the set $B(\dot{V}_1 = 0)$.

There is an analogous theorem for nonuniform asymptotic stability. But then the requirement of an infinitely small upper limit is imposed not on the main function V_1, but on the auxiliary function V_2, which thus possesses the following property.

Definition 6.1. The function $V_2(x, t)$ admits an infinitely small upper limit in the set $E(V^* = 0)$ if it is bounded, $V_2(x, t) = 0$ for $x \in E(V^* = 0)$, $t \in [0, \infty)$ and for any arbitrarily small number $l > 0$ and any region A_a there exists a neighborhood of the set $A_a \cap E(V^* = 0)$ in A_a in which $|V_2(x, t)| < l$ for $t \in [0, \infty)$.

If $V^*(x)$ is assumed negative definite in the above theorem, the auxiliary function V_2 is not needed, and we obtain a one-function criterion for asymptotic stability which generalizes Lyapunov's criterion in two directions. First, the requirement of an infinitely small upper limit is eliminated, and second, the definiteness requirement for the derivative $\dot{V}_1$ is somewhat relaxed.

This method of generalizing the Lyapunov criterion is due to Krasovskii /24, 12/, V.P. Marachkov /47/, Zubov /9/ and Reissig /25/.

Both the above two-function asymptotic stability theorems and the theorem of Barbashin and Krasovskii may be extended to the case of global asymptotic stability, if the conditions of the theorem include an estimate of the region of attraction of the unperturbed motion, and also to the case of asymptotic stability in the large, if the requirement of an infinitely great lower limit of the function V_1 is introduced.

We shall formulate a nonuniform asymptotic stability theorem (including the above-mentioned estimate), defining the properties of the functions only in some absolute sector Δ. We shall obtain a modification, on the one hand, of Kuz'min's theorem, and on the other, of the theorem of Barbashin and Krasovskii.

Theorem 6.2. Suppose there exist an absolute sector Δ and functions $V_1(x, t), V_2(x, t)$ which possess the following properties in $[\Delta]$.

1) $V_1(x, t)$ is positive definite and for some positive numbers λ, A $(0 < \lambda < A < h)$ satisfies the inequality $\sup [V_1(x, t)$ for $\{x, t\} \in [\Delta], \|x\| \leqslant \lambda] < \inf [V_1(x, t)$ for $\{x, t\} \in [\Delta], \|x\| = A]$;

2) $\dot{V}_1(x, t) \leqslant \varphi(t) V^*(x)$, where $V^*(x) \leqslant 0$ for $x \in (H)$, the function $\varphi(t)$ is positive in the mean;

3) $V_2(x, t)$ admits an infinitely small upper limit in the set $E(V^* = 0) \cap \cap [\bigcup_{\tau \in [0, \infty)} \Delta(\tau)]$;

4) $\dot{V}_2(x, t) \neq 0$ definitely in the set $E(V^* = 0) \cap [\bigcup_{\tau \in [0, \infty)} \Delta(\tau)]$,

Then the unperturbed motion $x = 0$ of system (1.1) is conditionally asymptotically stable for $t_0 \in [0, \infty)$, $x_0 \in \Delta(t_0) \cap E(\|x_0\| < \lambda)$. In this case, if system (1.1) is linear, the asymptotic stability is unconditional. In particular, when the absolute sector Δ is defined by an auxiliary function $W(x, t)$, we obtain a three-function theorem.

§ 7. Stability criteria using several functions V can be obtained on the basis of the theory of differential or integral inequalities, and also by fixed point principles.

A stability criterion with several functions V, based on the theory of differential inequalities, is suggested by the author in /17/. In /18/ it is generalized by replacing the requirement of stability of the auxiliary system by the property of quasi-stability in the following sense.

Given a system (4.1) and a function $\varphi(A)$, which is defined, continuous, and nondecreasing in the half-open interval $[0, H)$, where $\varphi(0) = 0$.

Definition 7.1. The half-line $y = 0$ $(t > 0)$ is called quasi-stable by virtue of system (4.1) with respect to $y_1, \ldots, y_l$ $(l \leqslant k)$ for the function $\varphi(A)$, if for any positive numbers $A < H$ and t_0 there exists a positive number $\alpha(A, t_0)$ $(\alpha < A, \alpha < \varphi(A))$ such that for any functions $\xi_1(t), \ldots, \xi_n(t)$ continuously differentiable for $t \in [0, \infty)$ for which $|\xi(t_0)| \leqslant \alpha$, any solution $y(t)$ of the system (1.2) with initial conditions $|y(t_0)| \leqslant \alpha$ is defined and satisfies the inequality $|y_1(t)| + \ldots\ldots + |y_l(t)| < \varphi(A)$ in any interval $[t_0, t^*]$ in which $|\xi(t)| \leqslant A$.

This definition generalizes the corresponding concepts introduced by Persidskii /26/, E.I. Dykhman /27/, and Zubov /9/, allowing for an observation of Erugin /7/ and a definition of Rumyantsev /28/.

Definition 7.1 is now transformed in the usual manner in order to obtain definitions of asymptotic, uniform, and conditional quasi-stability /18/.

Let us formulate a stability criterion with several functions V with respect to some of the variables $x_1, \ldots, x_m$. We put $H = \infty$, i.e., here the region Γ is

$$|x| < \infty, t > 0.$$

Theorem 7.1. Suppose there exist functions $V_1(x, t), \ldots, V_k(x, t)$ with the following properties in Γ:

$$1)\quad V_1(x, t) > 0, \ldots, V_l(x, t) > 0 \quad (l \leqslant k), \tag{7.1}$$

$$|V_1(x, t)| + \ldots + |V_l(x, t)| > V^*(x_1, \ldots, x_m) \quad (m \leqslant n),$$

where the function $V^*(x_1, \ldots, x_m)$ is positive definite with respect to $x_1, \ldots, x_m$;

2) $\dot{V}_s(x, t) \leqslant f_s(V_1(x, t), \ldots, V_k(x, t), x_1, \ldots, x_m, t)$ $(s = 1, \ldots, k)$;

3) each function $f_s(V_1, \ldots, V_k, x_1, \ldots, x_m, t)$ is nondecreasing in $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$;

4) the half-line $y = 0$ is quasi-stable (resp. asymptotically quasi-stable) by virtue of the system $dy_s/dt = f_s(y_1, \ldots, y_k, \xi_1(t), \ldots\ldots, \xi_m(t), t)$ $(s = 1, \ldots, k)$ with respect to $y_1, \ldots, y_l$ for the function $\varphi(A) = \inf [V^*(x_1, \ldots, x_m)$ for $x_1^2 + \ldots + x_m^2 > A^2]$ and the conditions $y_{10} > 0, \ldots\ldots, y_{l0} > 0$.

Then the unperturbed motion $x = 0$ of the system (1.1) is stable (resp, asymptotically stable) with respect to some of the variables $x_1, \ldots\ldots, x_m$.

If in this case the functions $V_1, \ldots, V_k$ admit infinitely small upper limits, and the quasi-stability stipulated in (4) is uniform with respect to t_0 (resp. the asymptotic quasi-stability is uniform with respect to y_0, t_0, ξ), then the stability of the unperturbed motion with respect to some of the variables is also uniform with respect to t_0 (resp. the asymptotic stability is uniform with respect to x_0, t_0).

If we have exact equalities $\dot{V}_s(x, t) = f_s(V_1(x, t), \ldots\ldots, V_k(x, t), x_1, \ldots, x_m, t)$ $(s = 1, \ldots, k)$ in conditions (2), and the quasi-stability in (4) is unconditional, then conditions (3) and (7.1) may be omitted.

When the functions f_s do not depend explicitly on x, the requirement of quasi-stability is replaced by the requirement that the auxiliary system be Lyapunov stable with respect to some of the variables $y_1, \ldots, y_l$ for $y_{10} > 0, \ldots\ldots, y_{l0} > 0$.

In particular, by putting $k = l = 1$, $m = n$, we obtain the theorems of Corduneanu /29/, which in turn generalize the classical theorems of Lyapunov and the modifications proved by Persidskii /30, 4/, Ibrashev /19/, I.G. Malkin /31/, Krasovskii /24/, Massèra /8/, Zubov /9/, Mel'nikov /14/.

Applications of differential and integral inequalities and fixed-point principles to stability problems and problems of the qualitative theory of differential equations have been studied by many authors, for example /32–37; 39, 41–45/.

In these studies the norm of the solution (sometimes the generalized norm or seminorm) or the norm of the right-hand sides of the differential equations of the perturbed motion is restricted directly by the corresponding inequalities. This yields interesting results for various specific types of systems.

M.A. Krasnosel'skii and Ya.D. Mamedov /49/, Opial /46/, M.A. Krasnosel'skii /40/, La Salle and Lefschetz /23/ use inequalities to estimate the function V or some functional. In Z.B. Tsalyuk's note /51/, where the idea of applying the conditions for solvability of Chaplygin's problem to the qualitative theory of equations is formulated in a very general form, the inequality is satisfied by an operator which maps the set of solutions into a certain partially ordered set; see also /40/.

The idea of applying Chaplygin's theory of differential inequalities directly to generalize Lyapunov's second method while retaining a single function V is due to La Salle /23/ and Corduneanu /29/. The most general results were obtained by Corduneanu (the core of his arguments are the theorems cited above) and Lakshmikantham /56—58/.

Theorem 7.1 also yields stability criteria which use higher derivatives of the function V; see, e.g., /17/.

Theorem 7.2. If there exists a positive definite function $V(x, t)$ whose second derivative $\ddot{V}(x, t) \leqslant p(t)\dot{V}(x, t)$, where $p(t)$ is continuous and

$$\int_{t_0}^{\infty} \exp\left[\int_{t_0}^{t} p(\tau)\, d\tau\right] dt < \infty$$

(in particular, if $p = \mathrm{const} < 0$), then the unperturbed motion is stable.

As is well-known, the method of Lyapunov functions is being increasingly applied to qualitative studies of differential equations /38, 50/. V.V. Nemytskii has drawn attention to the problem of global qualitative investigation of the method of Lyapunov functions and has surveyed the relevant literature /53/. In particular, P.N. Papush /54/ and M.B. Kudaev /55/ have obtained criteria which sometimes make it possible to visualize the type of system of integral curves using the properties of the function V and its first, second and third derivatives. In the simplest cases, these properties are of the same character as in Chetaev's first instability theorem (if one returns to the local problem).

Theorems 7.1, 7.2, and the like suggest other criteria for the behavior of integral trajectories.

Finally, let us formulate a criterion for the positiveness of the eigenvalues of the solutions of a linear system with continuous and bounded coefficients,* generalizing Chetaev's criterion /2/ ($k = 1$).

Theorem 7.3. Suppose there exist quadratic forms $V_1(x, t), \ldots, V_k(x, t)$ with continuous and bounded coefficients, having the following properties in Γ:

1) $V_1(x, t) > 0, \ldots, V_l(x, t) > 0$ $(l \leqslant k)$ and $V_1(x, t) + \ldots + V_l(x, t)$ is positive definite;

2) $\dot{V}_s(x, t) \leqslant p_{s1} V_1(x, t) + \ldots + p_{sk} V_k(x, t)$ $(s = 1, \ldots, k)$;

3) $p_{sr} > 0$ $(s \neq r)$, $p_{ss} < 0$, $p_{sr} = \mathrm{const}$ $(s, r = 1, \ldots, k)$;

4) all the roots of the equations $|p_{sr} - \lambda\delta_{sr}|_1^k = 0$ have negative real parts.

Then the unperturbed motion is exponentially stable, and all the eigenvalues of the solutions are positive.

In conclusion we note that the prospects for new stability criteria using several functions V and higher-order derivatives are apparently far from exhausted. We mention several directions for further investigation.

Stability criteria may be based on the distribution, topologically speaking, of the integral curves of the auxiliary system in the space $\{V_1, \ldots, V_k\}$ or $\{V, V, \ldots, V^{(k-1)},\}$ at least for $k = 2, 3$.

Different constructive stability criteria may be obtained as special cases of the theorems of §§ 4, 7.

Using various generalizations of inequality fixed-point theorems the given criteria may be modified.

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* Theorem 7.3 can be modified to yield an estimate of the eigenvalues of the solutions of a nonlinear system (1.1) with the aid of functions V which satisfy estimates characteristic for quadratic forms.

Material on the problems considered here can also be found in the books /59—63/, and in papers of Krasnosel'skii (/64/ and at the present conference).

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G.N. Mikishev and B.I. Rabinovich

SOME PROBLEMS IN DYNAMIC STABILITY ANALYSIS OF MECHANICAL SYSTEMS WITH DEFORMABLE ELEMENTS

1. Formulation of the problem

Consider a solid body containing a cavity partially filled with liquid, in a state of equilibrium or unperturbed motion; it is assumed that the field of mass forces of this motion has a potential. The body may be either rigid or elastic; in the latter case we shall assume that there is some axis of geometrical, mass and rigidity symmetry, whose direction coincides with that of the field of mass forces of the unperturbed motion. We shall study the stability of this motion for a given system of external forces, assuming that the generalized coordinates and velocities characterizing the motion are small. In general this does not imply the possibility of linearization with respect to all the generalized coordinates, since the existence of small nonlinearities (such as backlash, insensitivity zones) is permitted for the separate coordinates. The liquid is assumed to be of low viscosity, in the sense that the Reynolds number characterizing its perturbed motion is large. In other respects, we adopt the usual premises of the theory of small wave motions and of the linear theory of elasticity.

The authors undertook an adequate description of this mechanical system with a view to studying both stationary and nonstationary perturbations of the motion in a finite time interval, employing theoretical and experimental methods, digital and analog electronic computers, and physical models. It was therefore necessary to represent the equations of the perturbed motion in a definite form. The form most suitable is a system of ordinary differential equations of finite order, which is nearly canonical in the sense that the nondiagonal terms of the coefficient matrix are small. Stability problems may then be attacked by various approximate methods using algorithms easily realized on digital computers. Moreover, such systems are also convenient for the study of perturbed motion on analog computers, by virtue of the easy communication with the equipment and the proximity of the frequencies of the system to the corresponding partial frequencies. Finally, by employing equation systems with almost diagonal coefficient matrices the generalized coordinates, while their intuitive physical meaning is maintained, may be interpreted as quasi-principal coordinates of the mechanical system in question.

2. General framework for the solution of stability problems

1. Set up the general partial integro-differential equations and ascertain criteria of dynamic similitude.
2. Select appropriate partial mechanical systems and construct orthogonal basis systems, using the eigenfunctions of the corresponding boundary-value problems, which represent the normal modes of oscillation.
3. Use Galerkin's method to reduce the original partial equations to a system of ordinary differential equations; then reduce the latter to a system of order as low as possible.
4. Determine the coefficients of the systems, in particular, the partial frequencies and the added masses.
5. Investigate the local dynamic stability of the system at discrete points.
6. Investigate the stability of the system in a finite time interval.

This program is carried out on the basis of theoretical and experimental methods, which may sometimes duplicate or complement each other.

Let us briefly consider the methods used.

Selection of the partial systems is a fairly complicated problem, since inapt selection leads to strong interaction between the systems, and the corresponding generalized coordinates are not close to the principal coordinates of the system as a whole.

In particular, it is usually expedient to choose as partial mechanical systems a liquid in a motionless cavity with rigid walls, and a solid body (rigid or elastic) with a liquid whose free surface is perpendicular to the field of mass forces of the unperturbed motion. The first system of basis functions is then the set of the normal modes of oscillation of the liquid for wave motions at the free surface, and the second system is the set of oscillation modes for an elastic body with a quasi-frozen liquid. Both the oscillation modes and the corresponding partial frequencies can be calculated theoretically and determined experimentally. At the same time it is useful to determine the damping coefficients reduced to the corresponding partial frequencies. It should be emphasized that the only adequately reliable approach to this problem is the experimental one; this is our approach in the present paper, for which certain special methods were required.

Experimental investigation of the modes and frequencies of the natural oscillations of the liquid also makes it possible to determine the limits of applicability of the linear theory and to introduce corrections for small nonlinear effects.

Both the reduced damping coefficients of the oscillations of the liquid and their experimental values were determined by both theoretical and semiempirical methods in the presence of comparatively small dissipative effects; influence of the latter on the oscillation frequencies and modes is thus negligible and the resulting motion of an ideal liquid may be studied independently.

Once the basis functions are known, there is no essential difficulty in setting up the equations by Galerkin's method, but for high-order systems this operation becomes very tedious; special computer programs were prepared for this purpose.

The problem of reducing the complete system to one of lower order is complicated and does not have a universal solution. It is usually solved on the basis of the physical nature of the problem.

The coupling coefficients (added masses) were also determined on physically similar models, developed for this specific purpose by an experimental method.

The main difficulty in studying the stability of motion of the mechanical systems in question over a finite time interval is connected with the high order of the corresponding equations of the perturbed motion (up to order 40 and higher) and with the need, in a number of cases, for complete information on the character of the perturbed motion, including the frequencies and amplitudes of auto-oscillations. The usual stability methods are not effective for the solution of these problems even if modern high-power computers are used, and in a number of cases they are simply unsuitable. Therefore special methods had to be worked out for studying both local stability at discrete points and stability over a finite time interval.

All these methods are based on the specific structure assigned to the equations of the perturbed motion (small nondiagonal elements in the coefficient matrix) and the physical features of the system in question (small dissipative forces and harmonic linearization of the nonlinear sections).

As a result, we have developed methods which permit the effective solution of stability problems on digital and analog computers. A brief description of these methods will be given below.

We particularly wish to emphasize the possibilities of solving stability problems in complex mechanical systems with deformable elements by the use of analog devices, in which the mechanical system is represented by a physically similar model and the known external forces by special force generators. The work program of the latter is given by an analog computer whose input consists of instructions from the position and velocity sensors of the elements of the physical model. Using this method one can, in principle, take into account all the features of the actual mechanical system, including the most subtle dissipative and nonlinear effects in the most complete manner.

This method was extensively employed in carrying out the present work. The body-liquid system was represented by a physically similar model, and the external forces were determined by the analog part of the complex.

§ 3. Some results of investigations into the dynamics of liquid-filled rigid bodies

We present some results concerning the frequencies of the natural oscillations, the damping coefficients and added masses, which characterize the perturbed motion of a liquid partially filling moving cavities of different configurations with absolutely rigid walls.

We obtained theoretical solutions by separation of variables, variational methods and methods of long-wave theory (in cases of small relative depth).

In so doing, our main interest was in reliable determination of the added masses, with preassigned accuracy. For this purpose systems of coordinate functions (usually harmonic functions satisfying the maximum possible number of boundary conditions) were specially chosen, with regard to the specific form of the cavity.

For a check, the solution was usually duplicated, using an auxiliary system of basis functions. The convergence and stability of the machine computations were specially studied, and approximate solutions used for checking purposes. The frequencies and damping coefficients were determined experimentally by the method of free and forced oscillations. The added masses were determined experimentally by a specially developed technique involving the partial frequencies, the frequencies of the system and the measured amplitudes of the oscillations of the liquid.

As an example we cite theoretical and experimental results for cylindrical cavities with radial and coaxial partitions, spherical, conical, and toroidal cavities. There is complete agreement between the experimental and theoretical results, obtained for cylinders by the method of separation of variables, and for other cavities by the variational method. There is also good agreement with results obtained for some special cases by other authors.

4. Some results of investigations into the dynamics of liquid-filled elastic bodies

Given below are the results of theoretical and experimental investigations of longitudinal and transverse oscillations of thin-walled elastic cylinders and spherical thin-walled shells filled with liquid.

The theoretical results were obtained by rigorous solution of the hydrodynamical problems by separation of variables and in some cases by reducing the boundary-value problem to an integral equation, followed by calculation of the modes and frequencies of the shell oscillations by the Bubnov-Galerkin method.

The experimental results were obtained by studying physically similar models. Our main interest was in the oscillation frequencies, and we carried out a qualitative investigation of the oscillation modes for different numbers of nodes along the periphery.

Comparison of the theoretical and experimental results shows satisfactory agreement, except for cases of a large number of half-waves along the periphery (6—7), which display appreciable discrepancies requiring additional analysis.

For specific shells results are given, as examples, for cases of full, half-full and dry shells. The agreement between theory and experiment is quite satisfactory in all these cases.

5. Investigation of the stability of unperturbed motion over a finite time interval

We present some results of investigations of the perturbed motion of mechanical systems in question over a finite time interval, performed on a digital computer.

In the first stage we consider local stability by frequency and algebraic methods, based on investigation of the complex eigenvalue spectrum. In solving these problems the specific structural features mentioned above were used for the high-order equations of the perturbed motion. In a number of cases this yields simple geometrical forms for the elements of the gain-phase plot of an open-loop system, and simple approximate formulas for the roots of the characteristic equation of an open- and closed-loop system, based on an iteration method.

In the next stage we solve the problem of complete integration of the system of equations of the perturbed motion over a finite time interval on a digital computer, taking the principal nonlinearities into account.

To this end we effect a real transformation of the coefficient matrix for even derivatives, which leads to block diagonalization. In so doing the terms of the equations involving odd derivatives and nonlinear terms are assumed to be multiplied by a small parameter.

All these terms are transferred to the right-hand side and transformed appropriately in passing to the new generalized coordinates.

Essentially, we are realizing here A.I. Lur'e's idea of canonical transformations generalized to a special case of equations with slowly varying coefficients. Employing this method on a digital computer, we can carry out, with reasonable accuracy, diagonalization of a system of any order up to 40, actually performing this operation for a system of half the order; the corresponding transformations are real and cope with the case of real and complex roots, including also multiple ones.

The final stage is to apply the method of harmonic balance to each of the resulting independent subsystems with nonlinear terms of order ε (degenerating in certain cases to second-order equations).

The equations for the envelopes are integrated on a digital computer by a numerical method, allowing for variable coefficients in the general case.

To summarize, we can now envisage the most complete investigation of the dynamical properties of the mechanical systems in question over a finite time interval.

Yu. I. Neimark and N. A. Fufaev

EQUATIONS OF MOTION AND THE ROLE OF SMALL PARAMETERS IN THE DYNAMICS OF NONHOLONOMIC SYSTEMS

I. Equations of motion of nonholonomic systems

After Newton's discovery of the laws of mechanics, no essential difficulties were involved in setting up the equations of motion for a system of material points. However, the problem of eliminating the unknown reactions of the contraints then arose. For holonomic systems with ideal constraints, the problem was solved by Lagrange. For nonholonomic systems it was first solved by S.A. Chaplygin /1/. In 1895 Chaplygin proposed equations which seemed fundamental for analytical mechanics of nonholonomic systems:

$$\frac{d}{dt}\frac{\partial (T)}{\partial \dot{q}_s} - \frac{\partial (T)}{\partial q_s} + \frac{\partial V}{\partial q_s} + \sum_{i=1}^{m} \frac{\partial T}{\partial \dot{q}_i}\left[\sum_{k=m+1}^{n}\left(\frac{\partial B_k^{(i)}}{\partial q_s} - \frac{\partial B_s^{(i)}}{\partial q_k}\right)\dot{q}_k\right] = 0, \tag{1}$$

where $T = T(q_{m+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m, \dot{q}_{m+1}, \ldots, \dot{q}_n)$ is the kinetic energy of the system, $V = V(q_{m+1}, \ldots, q_n)$ is the potential energy. $B_k^{(i)}$ are the coefficients in the relations

$$\dot{q}_i = B_k^{(i)}(q_{m+1}, \ldots, q_n)\,\dot{q}_k \begin{pmatrix} i = 1, 2, \ldots, m \\ k = m+1, \ldots, n \end{pmatrix}, \tag{2}$$

which are the transformed equations of the nonholonomic constraints

$$A_1^{(i)}(q_{m+1}, \ldots, q_n)\,\dot{q}_1 + \ldots + A_n^{(i)}(q_{m+1}, \ldots, q_n)\,\dot{q}_n = 0. \tag{3.3}$$

The function $(T) = T(q_{m+1}, \ldots, q_n, B_k^{(1)}\dot{q}_k, \ldots, B_k^{(m)}\dot{q}_k, \dot{q}_{m+1}, \ldots, \dot{q}_n)$.

Although equations (1) are valid for a limited class of nonholonomic systems (the so-called Chaplygin systems), they actually describe almost all known nonholonomic systems.

In 1901 P.V. Voronets /2/ obtained the equations of motion of a nonholonomic system for the more general case in which T, V and $B_k^{(i)}$ may depend on all the generalized coordinates as well as on time. For Chaplygin systems Voronets' equations coincide with (1). In 1898 V. Volterra /3/ proposed equations of motion in kinematic characteristics, later to be called quasi-velocities. It should be noted that quasi-coordinates were actually used (though without proper justification) by Chaplygin /1/ in solving the problem of plane nonholonomic motion. In /4/ one of the authors derived conditions for the validity of Chaplygin's equations in the case in which not all the generalized coordinates are true, generalized them to the case of quasi-coordinates. Volterra's equations are nevertheless correct, despite the fact that his methods were wrong. In 1901 G. Maggi /6/ derived the equations of motion in a form which yields Volterra's equations. On the basis of Maggi's equations, one can introduce an acceleration function $S = \frac{1}{2} \sum_{i=1}^{N} m_i (\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2)$, express it in terms of generalized coordinates, and obtain P. Appell's equations $\frac{\partial S}{\partial \ddot{q}_s} = Q_s$; $(s = 1, 2, \ldots, n - m)$, which Appell himself derived in 1899 on the basis of the principle of least constraint /7/. L. Boltzmann /8/ (1902) and G. Hamel /9/ (1904) established a form of equations in quasi-coordinates which coincides for holonomic systems, referred to moving coordinates axes, and for nonholonomic systems. Let the quasi-velocities be related to the generalized velocities by (in general) nonintegrable relations $\omega_l = \alpha_{rs} (q_1, \ldots, q_n) \dot{q}_s$; $\dot{q}_\lambda = \beta_{\lambda r} (q_1, \ldots, q_n) \omega_r$; $(\alpha_{rs}\beta_{s\lambda} = \delta_{r\lambda})$. Then the Boltzmann-Hamel equations have the form

$$\frac{d}{dt}\left(\frac{dT^*}{\partial \omega_r}\right) + \sum_{s=1}^{n} \sum_{l=1}^{n} \gamma_{rsl} \omega_l \frac{\partial T^*}{\partial \omega_s} - \frac{\partial T^*}{\partial \pi_r} = \Pi_r \quad (r = 1, 2, \ldots, n), \tag{4}$$

where

$$\gamma_{rsl} = \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} \beta_{\lambda r} \beta_{\mu l} \left(\frac{\partial \alpha_{\lambda s}}{\partial q_\mu} - \frac{\partial \alpha_{\mu s}}{\partial q_\lambda} \right),$$

$$T^* = T (q_r, \beta_{\lambda r} \omega_r)$$

and $\frac{\partial T^*}{\partial \pi_r} = \sum_{\lambda=1}^{n} \beta_{\lambda r} \frac{\partial T^*}{\partial q_\lambda}$. For nonholonomic systems m quasi-coordinates are functions whose time derivatives are equal to the left-hand sides of the equations of the nonholonomic constraints (3). In this case (4) contains only $(n - m)$ equations.

Later versions of the equations were developed by J. Tzenoff /10/ (1924), G. Vränceanu /11/ (1926), J. Schouten /12/ (1929), and others using the equations of the nonholonomic constraints in different ways. If the initial d'Alembert-Lagrange equation is transformed as

$$\frac{d}{dt}\left(\frac{dT}{\partial \dot{q}_l} \delta q_l \right) - \delta T + \frac{\partial T}{\partial \dot{q}_l}\left(\delta \dot{q}_l - \frac{d}{dt} \delta q_l \right) = Q_l \delta q_l; \qquad (l = 1, 2, \ldots, n).$$

then everything depends on the specific commutation relations used (i. e., the value of the expression $\left(\delta \dot{q}_l = \frac{d}{dt} \delta q_l\right)$). In the language of geometry this means that the form of the equations of motion is determined by which curves in the configuration space are chosen as comparison curves for the trajectory of the real motion.

As shown in /13/, the problem reduces to some method of extending the definition of the operators d and δ outside the curve of motion (preserving their usual meaning on the curve itself). According to one method, the operators $d\delta$ and δd commute for all the generalized coordinates: the comparison curves are then kinematically impossible trajectories, and we obtain the Boltzmann-Hamel equations in quasi-coordinates. By another method, the operators $d\delta$ and δd commute only for independent generalized coordinates, and for dependent coordinates the expressions $\delta \dot{q}_l - \frac{d}{dt}\delta q_l$ depend on the equations of nonholonomic constraints. The comparison curves are then kinematically possible trajectories, and we obtain the equations of Voronets, Chaplygin, Maggi, and others, in true coordinates. This general approach yields another $(m-1)$ cases /14/ in which the equations of motion contain both quasi-coordinates and true coordinates: these cases are intermediate in form between the Boltzmann-Hamel and Voronets equations.

As early as the beginning of this century, it was remarked /15/ that the equations of motion of electromechanical systems containing distributed conductors with sliding contacts do not have the form of the Lagrange equations of the second kind. It was assumed that such systems are a variety of nonholonomic systems. In 1952 A. V. Gaponov showed /16/ that attaching a sliding contact to a conductor distributed either in a volume or over a surface is equivalent to imposing a nonholonomic constraint on the current distribution. Gaponov proposed new equations of motion for Chaplygin systems:

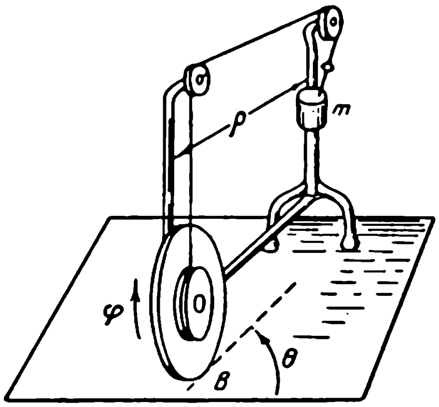
$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_r} - \frac{dL^*}{\partial q_r} - \frac{1}{\dot{q}_r}\sum_{s=m+1}^{n}\frac{\partial L^*}{\partial u_{rs}}\dot{u}_{rs} = Q_r \qquad (r = m+1, \ldots, u), \tag{5}$$

where the Lagrangian L^* is obtained from L by using the nonholonomic constraints (2), replacing the coordinates q_s in the coefficients $B_k^{(l)}(q_{m+1}, \ldots, q_n)$ by independent parameters u_{ks}, so that $B_k^{(l)} = B_k^{(l)}(u_{k,m+1}, u_{k,m+2}, \ldots\ldots, u_{kn})$. Thus $L^* = L^*(q_{m+1}, \ldots, q_n, \dot{q}_{m+1}, \ldots, \dot{q}_n, u_{m+1,m+1}, u_{m+1,m+2}, \ldots\ldots, u_{nn})$. . After carrying out the operations indicated in equations (5) one must put $u_{rs} = q_s$. It is advisable to use Gaponov's equations when the number of generalized coordinates is very large and the number $(n-m)$ of degrees of freedom small. This is precisely the situation in the general dynamics of electrical machines.

In a discrete description of an electrical machine with drum winding, the electrical state is characterized by a countable set of generalized coordinates. The commutator is a device which makes it possible to fix the current distribution in the revolving rotor in the coordinate system relative to the brushes. Thereby a countable set of nonholonomic constraints is imposed on the electrical coordinates, and as a result the system has a finite, usually small, number of degrees of freedom. The division of electrical machines into those with or without commutators corresponds to their division into holonomic and nonholonomic systems: commutator machines are Chaplygin systems.

II. The role of small parameters in the dynamics of nonholonomic systems

In this section we study the influence of small parameters on the dynamics of nonholonomic systems in a number of concrete examples. This helps to clarify the problem of the idealization involved in the neglect of small physical parameters when constructing a mathematical model of a nonholonomic system. As is well-known, an indication of the admissibility of some specific idealization can be inferred not only from a comparison of theoretical results with experimental data, but also from a comparison of the results of two different theories, only one of which utilizes the given idealization. The examples we present are quite varied, but they are all related by a common approach based on the division of the phase space into subspaces of slow and rapid motions. When the slow motions are stable relative to the rapid motions, the small physical parameters in question may safely be neglected. In the opposite case, the role of the small parameter is important.



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As a first example we consider a system with a nonlinear nonholonomic constraint.

The derivation of the equations of motion of mechanical systems with nonlinear nonholonomic constraints has been studied extensively /17—22/. At the time, Appell /18/, P. Delassus /23/, H. Beghin /24/, and others hotly debated whether it was possible to realize nonlinear nonholonomic constraints; however, these studies essentially contain no examples of systems with nonlinear ideal nonholonomic constraints differing significantly from the example given by Appell /18/ in 1911. This example was considered later by Hamel /20/, who set up its equations of motion by two methods. First, on the basis of the general equations of motion for systems with nonlinear constraints, Hamel obtained equations of the following form for the motion of a system by inertia

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = 0, \quad (a^2 + b^2)\ddot{z} = -gb^2 \tag{6}$$

and the equation of a nonlinear nonholonomic constraint

$$\dot{x}^2 + \dot{y}^2 = \frac{a^2}{b^2}\dot{z}^2. \tag{7}$$

Introducing the angles θ and φ by $\dot{x} = a\dot{\varphi}\cos\theta$; $\dot{y} = a\dot{\varphi}\sin\theta$; $\dot{z} = b\dot{\varphi}$, (6) then becomes the system

$$\dot{\theta} = 0, \quad (a^2 + b^2)\ddot{\varphi} = -gb, \tag{8}$$

and equation (7) becomes an identity. Hamel then arrived at the same equations (8) by letting ($\rho \to 0$) in the system

$$\rho(\rho\ddot{\theta} + a\dot{\theta}\dot{\varphi}) = 0, \quad (a^2 + b^2)\ddot{\varphi} - a\rho\dot{\theta}^2 = -gb, \tag{9}$$

which describes the motion of a nonholonomic system (Figure 1) with ordinary linear constraints. The agreement between the two systems of equations convinced Hamel of their correctness. However, since in passing to the limit the order of the system of differential equations (9) is reduced, i.e., degeneration occurs, it is not a priori clear whether the limit motions of the nondegenerate system coincide with those of the generate system. The present authors showed in 1963 /25/ that these motions are different.

Thus, a more rigorous approach to the study of the system in the Appell-Hamel example leads to motions which cannot be described by Hamel's equations.*

In fact, let us introduce new variables ξ and η defined by $\dot\theta = \alpha\xi$; $\dot\varphi = \beta\eta$, where $\alpha = \sqrt{1 + b^2/a^2}$; $\beta = gb/(a^2 + b^2)$. Denoting the physical parameter of the system by $\mu = |\rho|/\alpha\beta > 0$, we write equations (9) in the form

$$\mu\dot\xi = \mp\xi\eta, \quad \dot\eta = -1 \pm \mu\xi^2, \tag{10}$$

where the upper sign refers to the case $\rho > 0$, and the lower to the case $\rho < 0$. The motion of the system under discussion can be compared to a displacement of the representative point in the (ξ, η) plane. To delineate the trajectories in the (ξ, η) plane, we divide the second equation of (10) by the first and after integrating the resulting equation (in which the variables are separable) we obtain a family of integral curves

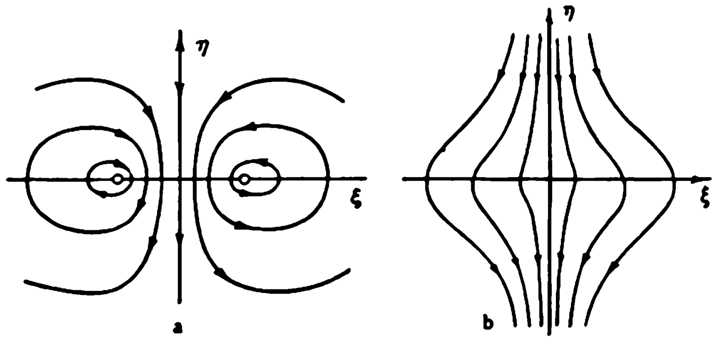
$$\eta^2 = \mu(\pm \ln \xi^2 - \mu\xi^2) + \text{const}, \tag{11}$$

shown in Figure 2. Consider the case $\rho > 0$. The motion of the representative point in the (ξ, η) plane is then described by the equations

$$\mu\dot\xi = -\xi\eta, \quad \dot\eta = -1 + \mu\xi^2. \tag{12}$$

Expressing the arbitrary constant in equation (11) in terms of the initial values $\xi = \xi_0$, $\eta = \eta_0$, we obtain

$$\eta^2 = f(\xi^2), \quad f(\xi^2) \equiv \mu\left[\ln \frac{\xi^2}{\xi_0^2} - \mu(\xi^2 - \xi_0^2)\right] + \eta_0^2. \tag{13}$$



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* A similar situation obtains for the friction regulator in V. S. Novoselov's example.

Using (13), equations (12) give the function $\xi = \xi(t)$ in the form

$$\mp\int_{\xi_0}^{\xi}\frac{d\xi}{\xi\sqrt{f(\xi^2)}} = t - t_0, \qquad (14)$$

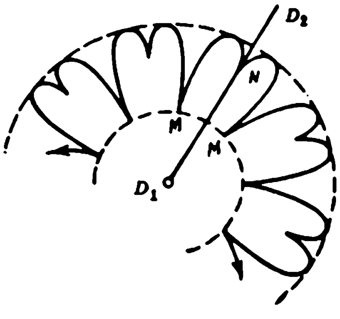
where the upper sign corresponds to the motion of the representative point along the upper half of the closed integral curve (Figure 2a) through the point (ξ_0, η_0), and the lower sign to its motion along the lower half. The angle θ of the plane of revolution of the wheel and the angle φ of its own revolution are, in terms of the variable ξ,

$$\theta - \theta_0 = \mp\mu\alpha\int_{\xi_0}^{\xi}\frac{d\xi}{\sqrt{f(\xi^2)}}, \quad \varphi - \varphi_0 = \mp\mu\beta\ln\frac{\xi}{\xi_0}. \qquad (15)$$

The absolute value of the velocity $\bar{v}_B$ of the point B of contact between the wheel and the plane (Figure 1) is

$$v_B = a\,|\dot{\varphi}| = a\beta\sqrt{f(\xi^2)}.$$

Hence it follows that when ξ assumes the values $\xi = \xi_1$ and $\xi = \xi_2$, where ξ_1 and ξ_2 are the roots of the equation $f(\xi^2) \equiv \mu\,[\ln(\xi^2/\xi_0^2) - \mu(\xi^2 - \xi_0^2)] + \eta_0^2 = 0$ (the condition for intersection of the integral curve with the axis $\eta = 0$), the point B halts momentarily. Since we then have $\dot{\theta} \neq 0$ and φ changes sign, the trajectory of the point B forms a "beak-shaped" curve with a cusp. Between these singular points the trajectory is a segment of a spiral curve. Thus, in one revolution of the representative point around a closed curve of Figure 2a, the point B describes a curve which we call a cell. One of these cells is marked $M - N - M$ in Figure 3: the point M corresponds to $\xi = \xi_1$ and the point N to $\xi = \xi_2$. A complete trajectory consists of identical $M - N - M$ cells along an annular strip, since each cell is symmetric with respect to the line D_1D_2 (Figure 3). Depending on the initial conditions and the relations between the parameters of the system, the trajectory of the point B is either a closed curve or a quasiperiodic curve everywhere dense in the annular region.



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When $\mu \to 0$ the whole (ξ, η) plane becomes a region of rapid motions (with respect to the coordinate ξ), with the exception of the line $\xi = 0$ which is the axis of slow motions. The half-line $(\xi = 0, \eta > 0)$ is then stable, and the half-line $(\xi = 0, \eta < 0)$ unstable relative to rapid motions. Let us find the limit motion of the system when $\mu \to 0$. To this end we consider the motion of the representative point in the (ξ, η) plane along one of the closed integral curves. The limit position (for $\mu \to 0$) of the integral curve is shown in Figure 4. In fact, it follows from equation (13) that when $\mu \to 0$

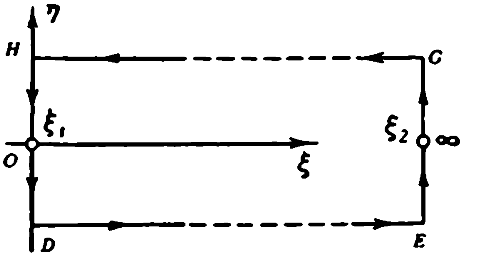
degenerates into a pair of straight lines $\eta = \pm\eta_0$. The roots ξ_1 and ξ_2 then tend to zero and to infinity, respectively, according to

$$|\xi_1| \approx e^{-\frac{\eta_0^2}{2\mu}}, \quad |\xi_2| \approx \frac{\eta_0}{\mu}.$$

Using these expressions in conjunction with (13)—(15), it is easy to calculate the variations of the time t and the angles θ and φ for the passage of the representative point through the segments $OD, DE, \ldots, HO$ in Figure 4. As a result we arrive at the following motion of the system for $\mu \to 0$, ($\rho > 0$): during time $t_1 = \eta_0$ the wheel rotates through an angle $\varphi_1 = -\beta\eta_0^2/2$, moving along a straight line $\theta_1 = \text{const}$, and then during time $t_2 = \eta_0/2$ rotates through the same angle $\varphi_2 = \varphi_1 = -\beta\eta_0^2/2$, and its plane turns through an angle $\theta_2 = \alpha_0$. The trajectory of the point B in this section has the form of an arc whose length is equal to that of the previous rectilinear segment. The plane of the wheel then turns instantaneously through an angle $\theta_3 + \theta_4 = 2\alpha$, and the direction of the intrinsic rotation of the wheel is reversed. The trajectory of the point B then forms a corner point with a finite angle $\theta = 2\alpha$. Then the wheel again describes an arc-like trajectory, rotating through an angle $\varphi_5 = \beta\eta_0^2/2$, during time $t_5 = \eta_0/2$ and its plane turns through an angle, $\theta_5 = \alpha$ Finally, during time $t_6 = \eta_0$, the wheel moves along a straight line, rotating through an angle $\varphi_6 = \beta\eta_0^2/2$. At the end of this segment of the trajectory, the direction of the wheel's intrinsic rotation is reversed, and the direction of the motion changes. The complete trajectory of the point B has the form of a rosette, either closed or an open curve everywhere dense in some annular region. The dimensions of this region are proportional to the initial value $|\eta_0|$ of the intrinsic angular velocity of the wheel.

It is easy to see that this limit motion differs from the motion of the system ($\rho = 0$). In fact, in the limit case, for $\mu \to 0$, the equations of motion (10) become $\xi = 0$; $\dot{\eta} = -1$.

Hence it follows that for any initial conditions, the motion of the system corresponds to a displacement of the representative point in the (ξ, η) plane along a straight line $\xi = 0$ at a constant velocity $\dot{\eta} = -1$. This means that the trajectory of the contact point of the wheel with the plane is always a straight line along which the wheel rolls with constant acceleration.



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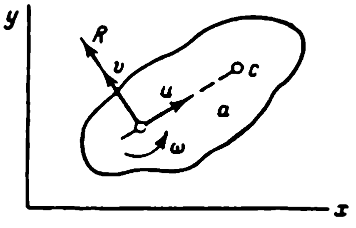
The case $\rho < 0$ is similar: the trajectory of the point B coincides qualitatively with that of the Chaplygin—Carathéodory sledge. For $\mu \to 0$ the form of the trajectory is the same, except that the cusp becomes a corner point.

As another example clarifying the role of the parameters neglected in the idealization of a real physical system, we mention the influence of the usually neglected area of contact of a sphere with a plane on the dynamics of a sphere rolling along a rough plane. Consider the inertial motion of a homogeneous heavy sphere over a horizontal rough plane and take into account that the contact surface of the ball with the plane is a small circle of radius a. The case $a = 0$ corresponds to point contact, which in the case of rolling without slipping leads to a nonholonomic constraint. The limit system for $a \to 0$ is a sphere with point contact. Here too it is not clear a priori whether the limit motion of the system for vanishingly small a $(a \neq 0)$ coincides with the motion of the limit system $(a = 0)$. A study of the dynamics of the sphere shows that they do coincide and, consequently, the idealization involved in replacing the surface of contact by a point is justified.



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Finally, we consider an example in which the theory of differential equations in which the leading coefficient is a small parameter which contributes to the solution of a certain problem concerning the realization of a nonholonomic constraint. In 1933 Carathéodory /27/ studied the dynamics of a Chaplygin sledge, in the special case when the center of mass of the sledge is situated on a straight line in the plane of the blade. The equations of motion of the sledge are

$$\dot{u} = a\omega^2, \quad ak^2\dot{\omega} = -u\omega, \tag{16}$$

where u is the velocity of the point of contact of the blade with the plane, ω is the angular velocity of rotation of the sledge, $k^2 = 1 + J/ma^2$ (Figure 5).

The reaction force R of the nonholonomic constraint prevents the sledge from slipping in a direction perpendicular to the plane of the blade, and therefore the velocity component in this direction is $v = 0$. Assuming R to have the character of viscous friction, Carathéodory represented this force in the form

$$R = -Nv,$$

where N—the coefficient of viscous friction—is very large. For $v \neq 0$ the equations of motion of the sledge are

$$v = \varepsilon a\dot{\omega}; \quad \dot{u} = a\omega^2 + \varepsilon a\omega\dot{\omega}; \quad ak^2\dot{\omega} + u\omega = -\varepsilon a\ddot{\omega}, \tag{17}$$

where $\varepsilon = J/Na^2$ is a small parameter. For $\varepsilon \to 0$, equations (17) coincide with (16). However, if we consider the motion for the initial conditions $u_0 = v_0 = 0$, $\omega_0 = kc$, where $c = \text{const} \neq 0$, equation (16) gives initial values $\dot{\omega}_0 = 0$, $\ddot{\omega}_0 = -kc^3$, and equations (17) the values $\dot{\omega}_0 = 0$, $\ddot{\omega}_0 = 0$. Hence Carathéodory concluded that the nonholonomic constraint in question cannot be realized by means of viscous friction forces.

A more careful investigation disproves Carathéodory's conclusion. In fact, let us write equations (17) in the form:

$$\dot{\omega} = \sigma; \quad \dot{u} = a\omega^2 + \mu\omega\sigma; \quad \mu\dot{\sigma} = -ak^2\sigma - u\omega, \tag{18}$$

where $v = \mu\sigma$; $\mu = a\varepsilon$ is a small parameter. The third equation of (18) implies that for $\mu \to 0$ the phase space (u, ω, σ) becomes a region of rapid motions, with the exception of the surface

$$ak^2\sigma + u\omega = 0, \tag{19}$$

which is a region of slow motions stable relative to the rapid motions.

In the case of small values of the parameter μ, $(\mu \neq 0)$, the representative point in the (u, ω, σ) space reaches the μ-neighborhood of the surface (19) from points outside a finite neighborhood of this surface in time of order $\mu \ln \frac{1}{\mu}$, and remains there /28/. Now let us study the behavior of the function $\dot{\sigma} = \dot{\sigma}(t)$. To this end we consider the motion of the representative point in the $(u, \omega, \dot{\sigma})$ space. Differentiating the third equation of (18) and eliminating the variable σ, we obtain from (18) a system of differential equations

$$\begin{gathered}
ak^3 \frac{du}{dt} = a^2k^3\omega^2 - \mu\omega(u\omega + \mu\dot{\sigma}), \\
ak^3 \frac{d\omega}{dt} = -u\omega - \mu\dot{\sigma}, \\
\mu ak^3 \frac{d\dot{\sigma}}{dt} = [\mu(u + \mu\omega^2) - a^2k^4]\dot{\sigma} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^2,
\end{gathered} \tag{20}$$

which describe the motion of the representative point in the $(u, \omega, \dot{\sigma})$ space. From the third equation of (20) it follows that when $\mu \to 0$ the $(u, \omega, \dot{\sigma})$ space becomes a region of rapid motions (with respect to the coordinate $\dot{\sigma}$), with the exception of the surface

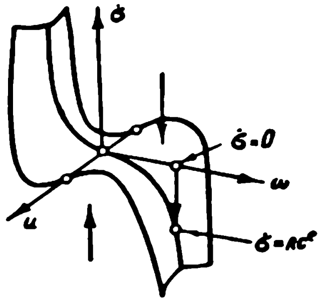
$$a^2k^4\dot{\sigma} = \omega u^2 - a^2k^2\omega^3, \tag{21}$$

which is a region of slow motions (Figure 6). These slow motions are stable relative to the rapid motions. For small values of the parameter μ, $(\mu \neq 0)$, the region of slow motions is the μ-neighborhood of the surface

$$[\mu(u + \mu\omega^2) - a^2k^4]\dot{\sigma} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^2 - 0. \tag{22}$$

In a bounded region of the $(u, \omega, \dot{\sigma})$ space the surfaces (22) and (21) differ by a small quantity of order μ, and when $\mu \to 0$, they coincide.

In the case of Carathéodory's initial conditions $(u_0 = 0, \omega_0 = kc, \dot{\sigma}_0 = 0)$, the representative point in the $(u, \omega, \dot{\sigma})$ space is situated outside a finite neighborhood of the surface (22) and, by what has been said above, reaches the μ-neighborhood of the surface in time of order $\mu \ln \frac{1}{\mu}$. When $\mu \to 0$ this time interval tends to zero. For values $u = 0$ and $\omega = k$ on the surface (21), we have $\dot{\sigma} = -kc^3$, and therefore when $\mu \to 0$ the change of $\dot{\sigma}$ from $\dot{\sigma} = 0$ to $\dot{\sigma} = -kc^3$ becomes instantaneous (see Figure 6). Thus, the contradiction which Carathéodory discovered is resolved as follows: although the values $\ddot{\omega} = \dot{\sigma}$ are indeed different at the initial moment $t = 0$ according to whether $\mu = 0$ or $\mu \neq 0$, $(\mu \ll 1)$, this difference disappears



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in a time interval of order $\mu \ln \frac{1}{\mu}$. Consequently, as $\mu \to 0$ the limit motion of the nondegenerate system (18) differs by an arbitrarily small amount from the motion of the degenerate system $(\mu = 0)$ /16/.

The above analysis implies that the nonholonomic constraint involved in the motion of a blade over a plane can be realized by means of viscous friction forces; it corresponds to the case of an infinite coefficient of viscous friction.

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V.S. Novoselov

ANALYTICAL DYNAMICS AND OPTIMUM TRANSFER

§ 1. Stationarity conditions

Suppose the equations of motion of an object are

$$\dot{x}_i = F_i(x_j, u_k, t) \quad (i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, r), \tag{1.1}$$

where the $u_k(t)$ are piecewise-continuous controls (control functions) and t is the time.

The controls must satisfy the conditions

$$u_{k1} \leqslant u_k \leqslant u_{k2}, \quad u_{k1} = \mathrm{const}, \quad u_{k2} = \mathrm{const}. \tag{1.2}$$

In particular, for all or some controls we may have $u_{k1} = -\infty$, $u_{k2} = \infty$.

We shall assume that every admissible control corresponds to a single trajectory of system (1.1) issuing from a point with coordinates $x_i^{t_1}$ at the initial instant t_1. The final instant will be denoted by t_f, the coordinates of the final point by $x_i^{t_f}$.

The boundary values of the variables are related in general by implicit conditions

$$R_\rho(x_i^{t_1}, x_i^{t_f}, t_1, t_f) = 0, \quad (\rho = 1, 2, \ldots, l \leqslant 2n + 2). \tag{1.3}$$

A trajectory satisfying conditions (1.1)—(1.3) will be called admissible.

The controls must be chosen so that the functional

$$\int_{t_1}^{t_f} F_0(x_i, u_f, t)\, dt + R_0(x_i^{t_1}, x_i^{t_f}, t_1, t_f), \tag{1.4}$$

be a minimum relative to its values for other admissible trajectories.

The functional (1.4) on admissible trajectories is equivalent to the following functional:

$$U = \int_{t_1}^{t_f} L\, dt + R, \tag{1.4'}$$

where

$$L = F_0 + \sum_{i=1}^{n} \lambda_i (\dot{x}_i - F_i), \quad R = R_0 + \sum_{\rho=1}^{l} \nu_\rho R_\rho.$$

Here the λ_i are explicit functions of time, to be determined, and the ν_ρ are undetermined constants.

The controls u_k may be replaced by new controls γ_k using formulas $u_k = u_k(\gamma_k)$, where the derivatives $du_f/d\gamma_f$ vanish at the boundaries of the region (1.2). Examples of this procedure are discussed in /1, 2/.

Alternatively, we need not express u_k in terms of γ_k, but consider an increased number of controls and introduce additional undetermined multipliers λ_{n+k} which are functions of time. We must then add the sum $\sum_{k=1}^{r} \lambda_{n+k} [u_k - u_k(\gamma_k)]$ to the function L.

The total variation of the criterion functional is given by the formula

$$\Delta \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} \Big[\sum_{k=1}^{r} \frac{\partial L}{\partial \gamma_k} \delta\gamma_k - \sum_{i=1}^{n} \Big(\dot{\lambda}_i + \frac{\partial H}{\partial x_i}\Big) \delta x_i\Big] dt + \Big(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t\Big)_{t_i}^{t_f}. \qquad (1.5)$$

Here we denote

$$H = \sum_{i=1}^{n} \lambda_i \dot{x}_i - L = -F_0 + \sum_{i=1}^{n} \lambda_i F_i. \qquad (1.6)$$

Let the functional (1.4) have a smooth minimum. By equating ΔU to zero, we then obtain necessary conditions for an extremum

$$\dot{\lambda}_i = -\partial H/\partial x_i, \quad (i = 1, 2, \ldots, n), \qquad (1.7)$$

$$\partial H / \partial \gamma_k = 0, \quad (k = 1, 2, \ldots, r), \qquad (1.8)$$

$$\Delta R + \Big(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t\Big)_{t_i}^{t_f} = 0. \qquad (1.9)$$

If the total variations Δx_i in (1.9) are expressed by the formulas $\Delta x_i = \delta x_i + \dot{x}_i \Delta t$ and the terms in Δt and δx_i equated to zero, then instead of (1.9) we obtain $2n + 2$ transversality conditions in Bolza's expanded form /13/.

We mention the frequently encountered time optimal control problem, when the criterion functional may not have a smooth minimum. In this case

$$F_0 = 1, \quad R_0 = 0, \quad \Delta \int_{t_i}^{t_f} L dt = \Delta t_f - \Delta t_i,$$

and formula (1.5) assumes the form

$$\int_{t_i}^{t_f} \Big[\sum_{k=1}^{r} \frac{\partial L}{\partial \gamma_k} \delta\gamma_k - \sum_{i=1}^{n} \Big(\dot{\lambda}_i + \frac{\partial H}{\partial x_i}\Big) \delta x_i\Big] dt + \Big[\sum_{i=1}^{n} \lambda_i \Delta x_i - (1 + H) \Delta t\Big]_{t_i}^{t_f} = 0.$$

Hence we obtain $H_{t_i} = H_{t_f} = -1$, equations (1.7) and (1.8) and the transversality condition

$$\Delta R + \left(\sum_{i=1}^{n} \lambda_i \Delta x_i\right)_{t_f}^{t_i} = 0, \quad R = \sum_{\rho=1}^{l} \nu_\rho R_\rho.$$

In the case of an increased number of controls, equations (1.8) are replaced by

$$\frac{\partial H}{\partial u_k} - \lambda_{n+k} = 0, \quad \lambda_{n+k} \frac{du_k}{d\gamma_k} = 0. \tag{1.8'}$$

Suppose the controls γ_k are not introduced, and the investigation involves only the initial controls u_k. Equations (1.8) must then be replaced by the conditions

$$\begin{gathered} \partial H/\partial u_k = 0 \quad \text{for} \quad u_{k1} < u_k < u_{k2}, \\ \frac{\partial H}{\partial u_k} \leqslant 0 \quad \text{for} \quad u_k = u_{k1}, \ \frac{\partial H}{\partial u_k} \geqslant 0 \quad \text{for} \quad u_k = u_{k2}. \end{gathered} \tag{1.8''}$$

Remark 1. The system (1.1) and (1.7) is a canonical system with Hamiltonian H, momenta λ_i and coordinates x_i. This canonical system differs from a mechanical canonical system in the linear dependence of H on the λ_i and the occurrence of additional variables u_k or γ_k.

Since system (1.1) and (1.7) is canonical and the limit controls u_{k1} and u_{k2} are constant, on any section of an optimal trajectory with continuous controls we have

$$H = h + \int_{t_0}^{t} \left(-\frac{\partial F_0}{\partial t} + \sum_{i=1}^{n} \lambda_i \frac{\partial F_i}{\partial t}\right) dt, \tag{1.10}$$

where $h = \text{const}$, t_0 is the initial instant of the section in question.

If the functions F_0 and F_i do not depend explicitly on t, H is constant on any section of this type.

Remark 2. A point of discontinuity of the controls is called a corner point. Let us vary the position and time of some corner point by quantities Δx_i and $\Delta\tau$; using formula (1.5) for an optimal trajectory on both sides of the corner point, the variation of the criterion functional U is

$$\left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t\right)_{\tau-0} - \left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t\right)_{\tau+0},$$

where

$$\Delta x_i|_{\tau-0} = \Delta x_i|_{\tau+0} = \Delta x_i, \quad \Delta t|_{\tau-0} = \Delta t|_{\tau+0} = \Delta\tau.$$

For an optimal trajectory the variation of the functional must vanish. Hence we obtain the well-known Weierstrass—Erdmann conditions for the continuity of λ_i and H along the whole optimal trajectory.

In the case of stationary functions F_0 and F_i, therefore, H is the same constant for any optimal trajectory. For example, for time optimal control it follows from the previous considerations that $H = -1$.

§ 2. Maximum principle

Integration of the variation of the criterion functional along the trajectory tubes is a method often used in analytical dynamics /4/. We shall use it to obtain necessary and sufficient extremum conditions.

Consider any admissible trajectory AB of system (1.1) and (1.7) with controls $\{\tilde{u}_k\}$. This trajectory passes through any two points A and B which satisfy equations (1.3) at some instants t_{i} and $\tilde{t}_{\mathrm{i}}$. In particular, the points A and B and the instants $\tilde{t}_{\mathrm{i}}$ and $\tilde{t}_{\mathrm{f}}$ may coincide with the initial and final positions and instants of the optimal trajectory, and the trajectory AB may differ from the optimal trajectory only in some intermediate interval in which the controls $\{u_k\}$ differ from the controls $\{u_k\}$ of the optimal trajectory.

In the $2n+1$-dimensional configuration space of the coordinates x_i, the momenta λ_i, and the time t, we pass a curve C°_{μ} through the point A and the initial point of the optimal trajectory:

$$x_i^{\mathrm{i}} = x_i^{\mathrm{i}}(\alpha_p),\ \lambda_i^{\mathrm{i}} = \lambda_i^{\mathrm{i}}(\alpha_p),\ t^{\mathrm{i}} = t(\alpha_p, v'\mathrm{i}),\ \alpha_p = \alpha_p(\mu)\quad (p = 1, 2, \ldots, 2n). \tag{2.1}$$

We note that for the time being formulas (2.1) may be arbitrary if the condition $\partial t/\partial v > 0$ is satisfied and the contour C^0_{μ} passes through the two indicated points. Additional restrictions on the choice of C^B_{μ} will be given below. We shall call any trajectory of system (1.1) and (1.7) an extremal. Through a point of the curve AB and the section of the curve C°_{μ} between the initial point of the optimal curve and the point A we now pass extremals which correspond to controls which coincide with $\{u_k\}$ on the optimal trajectory.

Suppose the relations (2.1) are such that for $v = v'\mathrm{f}$ the curve C°_{μ} is transformed by the indicated extremals into a curve C_{μ} through the terminal point of the optimal trajectory and the point B. The control C_{μ} is determined by the general formulas

$$\begin{aligned} x_i &= x_i(x_i^{t_\mathrm{i}}(\alpha_p),\quad u_k(t), t), \\ \lambda_i &= \lambda_i(\lambda_i^{t_\mathrm{i}}(\alpha_p), x_i^{t_\mathrm{i}}(\alpha_p), u_k(t), t), \\ t &= t(\alpha_p, v'\mathrm{f}),\quad \alpha_p = \alpha_p(\mu). \end{aligned} \tag{2.2}$$

Through the points of the section of the curve C°_{μ} between the initial point of the optimal trajectory and the point A, we pass extremals with controls $\{u_k\}$, such that these extremals, for $v = v'\mathrm{f}$, completely fill the section of the curve C_{μ} between the final point of the optimal trajectory and the point B, subject to the following conditions:

$$\begin{aligned} u'_k(t(\alpha_p(\mu), v'\mathrm{i})) &= u_k(t(\alpha_p(\mu), v'\mathrm{i})), \\ u'_k(t(\alpha_p(\mu), v'\mathrm{f})) &= u_k(t(\alpha_p(\mu), v'\mathrm{f})). \end{aligned}$$

At any instant t, $\{u_k\}$ and $\{\tilde{u}_k\}$ are boundary values for the controls $\{u_k'\}$. The last condition on the choice of the contour C_μ^0 and the controls $\{u_k'\}$ is:

$$\Delta R + \left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t\right)_{t(\alpha_p,\, v^{t_{\mathrm{i}}})}^{t(\alpha_p,\, v^{t_{\mathrm{f}}})} = 0. \tag{2.3}$$

All the above conditions can be satisfied, by virtue of the arbitrary dependence of $x_i^{t_{\mathrm{i}}}$ and $\lambda_i^{t_{\mathrm{i}}}$ on α_p, the functions α_p on the parameter μ, and u_k' on the time t, and the fact that the function $t(\alpha_p, v)$ is also arbitrary.

Let us integrate expressions (1.5) in the above field of extremals, whose controls coincide with the optimal controls $\{u_k\}$:

$$\begin{gathered} -\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} L dt = - \int_{C_\mu^0\, t_{\mathrm{i}}}^{\tilde{t}_{\mathrm{f}}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t + \\ + \int_{C_\mu\, t_{\mathrm{f}}}^{\tilde{t}_{\mathrm{f}}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t - \int_{AB\, \tilde{t}_{\mathrm{i}}}^{\tilde{t}_{\mathrm{f}}} \left(\sum_{i=1}^{n} \lambda_i \eta_i - H\right) dt. \end{gathered} \tag{2.4}$$

The integral on the left-hand side is calculated along the optimal trajectory, the first integral on the right-hand side along the curve C_μ^0 from the initial point of the optimal trajectory to the point A, the second integral along the curve C_μ from the final point of the optimal trajectory to the point B, and the third integral along the trajectory AB.

The quantities η_i are determined by formulas (1.1) in the form $\eta_i = F_i(x_j, \tilde{u}_k, t)$. The function H and the factors λ_i in the third integral on the right-hand side are calculated for the extremals of the field at their points of intersection with the curve AB. The controls occurring explicitly in H therefore coincide with the optimal controls $\{u_k\}$.

Integrating formula (2.3) in the field of the extremals with the controls $\{u_k'\}$, we obtain

$$\begin{gathered} \int_{C_\mu\, t_{\mathrm{f}}}^{\tilde{t}_{\mathrm{f}}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t - \int_{C_\mu^0\, t_{\mathrm{i}}}^{\tilde{t}_{\mathrm{i}}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t = \\ = R_0(x_i^{t_{\mathrm{i}}}, x_i^{t_{\mathrm{f}}}, t_{\mathrm{i}}, t_{\mathrm{f}}) - R_0(\tilde{x}_i^{\tilde{t}_{\mathrm{i}}}, \tilde{x}_i^{\tilde{t}_{\mathrm{f}}}, \tilde{t}_{\mathrm{i}}, \tilde{t}_{\mathrm{f}}). \end{gathered} \tag{2.5}$$

By virtue of (1.6), which defines the function H, we find

$$\sum_{i=1}^{n} \lambda_i \eta_i = H(x_i, \lambda_i, \tilde{u}_k, t) + L(x_i, \lambda_i, \tilde{u}_k, t). \tag{2.6}$$

Using formulas (1.4'), (2.4)—(2.6) and the fact that the optimal trajectory and the trajectory AB satisfy conditions (1.3), we have

$$U_{AB} - U_{\mathrm{opt}} = \int_{AB\, \tilde{t}_{\mathrm{i}}}^{\tilde{t}_{\mathrm{f}}} [-H(x_i, \lambda_i, \tilde{u}_k, t) + H(x_i, \lambda_i, u_k, t)]\, dt > 0. \tag{2.7}$$

Hence we obtain a necessary condition for the minimality of the criterion functional—that H be maximum on the optimal trajectory:

$$H(x_i, \lambda_i, u_k, t) > H(x_i, \lambda_i, \tilde{u}_k, t). \tag{2.8}$$

Condition (2.8) is the basic condition of L.S. Pontryagin's maximum principle /5/ and is equivalent to the Weierstrass condition /3/. From (2.8), in particular, follow conditions (1.8), (1.8') and (1.8'').

Suppose condition (2.8) is satisfied for any controls $\{\tilde{u}_k\}$, certain controls $\{u_k\}$, and x_i and λ_i solving equations (1.1) and (1.7) for any $x_i^{t_1}$ and $\lambda_i^{t_1}$ and for the controls $\{u_k\}$. Then if conditions (1.1)—(1.3), (1.7) and (1.9) are satisfied on some trajectory with controls $\{u_k\}$, then, as shown by the above considerations, the trajectory in question is optimal.

§ 3. Application of the Hamilton-Jacobi method

The optimal trajectory $\{x_i\}$ and the undetermined factors $\{\lambda_i\}$ satisfy canonical equations with Hamiltonian (1.6) and controls $\{u_k\}$, considered as explicit functions of time. This canonical system can be solved by the Hamilton-Jacobi method.

To this end, consider the partial differential equation

$$\frac{\partial W}{\partial t} + H\left(x_i, \frac{\partial W}{\partial x_i}, u_k(t), t\right) = 0. \tag{3.1}$$

By virtue of formula (1.6), equation (3.1) becomes

$$\frac{\partial W}{\partial t} + \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} F_i = F_0. \tag{3.1'}$$

Formula (3.1') shows that the Hamilton-Jacobi equation for the optimal control problem is a first-order linear partial differential equation.

Suppose that for certain controls $u_k(t)$ the complete integral of equation (3.1) has been found:

$$W = W(x_i, a_i, t) + a_0 \tag{3.2}$$

where a_i are n independent constants and a_0 is an additive constant. The solution of equations (1.1) and (1.7) for these controls $u_k(t)$ is then given by the formulas

$$\frac{\partial W}{\partial a_i} = b_i, \tag{3.3}$$

$$\frac{\partial W}{\partial x_i} = \lambda_i, \tag{3.4}$$

where b_i are new independent constants.

It is well-known that the complete integral of equation (3.1) is determined up to an additive constant by the action integral. Therefore

$$W = \int_{t_1}^{t} L\,dt + a_0.$$

The function R in formula (1.4) depends on the initial and final values of the variables and is a certain additive constant. Hence, the criterion functional U is equal to the complete integral of equation (3.1), up to an additive constant.

On the basis of formulas (3.1) and (3.4), for any explicit representation of the functional on trajectories of system (1.1) and (1.7) we have

$$\frac{\partial U}{\partial x_i} = \lambda_i, \quad \frac{\partial U}{\partial t} = -H. \tag{3.5}$$

Formulas (3.5) can be used in programming numerical solution of variational problems, for example, by the method of residuals /6/.

If $a_i = \lambda_i^{t_1}$ and $b_i = x_i^{t_1}$, the complete integral of the Hamilton-Jacobi equation has the form /7/

$$W(x_i, \lambda_i^{t_1}, t) = \int_{t_1}^{t} L\,dt + \sum_{i=1}^{n} x_i^{t_1} \lambda_i^{t_1}. \tag{3.6}$$

To illustrate the application of the Hamilton-Jacobi method, let us obtain a condition for optimal coplanar impulsive transfer between elliptical orbits in the gravitational field of a spherically symmetrical central body.

We consider the motion of a point in a system of polar coordinates (r, φ) with the origin at the central body. The equations of motion are

$$\left.\begin{aligned} &\dot v_r = v_\varphi^2 r^{-1} - k^2 r^{-2} + \dot\beta\cos\psi,\\ &\dot v_\varphi = v_r v_\varphi r^{-1} + \dot\beta\sin\psi,\\ &\dot r = v_r, \quad \dot\varphi = v_\varphi r^{-1}, \end{aligned}\right\} \tag{3.7}$$

where k^2 is a constant, v_r and v_φ are the velocity components of the point along the radius-vector and the transverse direction, $\beta = -u_r \ln m$, u_r is the constant effective exhaust velocity, m is the mass, ψ is the angle between the thrust direction and the radius-vector, measured in the direction from the radius-vector to the velocity of the point.

Neglecting terms of the order of the ratio of the longest propulsion period to the total flight time, the equations of impulsive changes in velocity for these sections are

$$\dot v_r = \dot\beta\cos\psi, \quad \dot v_\varphi = \dot\beta\sin\psi, \quad \dot r = 0, \quad \dot\varphi = 0. \tag{3.7'}$$

Based on the possibilities of rocket engineering and permissible overload, a restriction must be imposed on the control:

$$0 < \dot\beta < G, \quad G = \text{const}. \tag{3.8}$$

The equations of motion for the coasting periods can be obtained from system (3.7) by setting $\dot\beta = 0$.

Understanding optimum in the sense of least mass flow, the criterion functional is

$$\int_{t_i}^{t_f} \dot\beta\, dt = u_r \ln (m_i m_f^{-1}), \tag{3.9}$$

where m_i is the initial mass and m_f the final mass of the ship, t_i and t_f are the initial and final instants.

The functional (3.9) must be minimized subject to the constraints (3.7) or (3.7'), the restriction (3.8) on the control $\dot\beta$, and the boundary conditions /8/

$$v_r^i = k e_i p_i^{-\frac{1}{2}} \sin f_i, \quad v_\varphi^i = k p_i^{\frac{1}{2}} r_i^{-1}, \quad r_i = p_i (1 + e_i \cos f_i)^{-1},$$
$$\varphi_i = f_i + \omega_i, \tag{3.10}$$

$$v_r^f = k e_f p_f^{-\frac{1}{2}} \sin f_f, \quad v_\varphi^f = k p_f^{\frac{1}{2}} r_f^{-1}, \; r_f = p_f (1 + e_f \cos f_f)^{-1}, \; \varphi_f = f_f + \omega_f. \tag{3.11}$$

Here "i" refers to the initial orbit, and "f" to the final orbit. Other notations used in formulas (3.10) and (3.11): e is the eccentricity, p is a parameter, ω is the angular distance of the pericenter from the polar axis, f is the true anomaly.

In the case under consideration $R_0 = 0$, and the boundary conditions are expressed by the explicit relations (3.10) and (3.11). We therefore do not introduce the undetermined constants (3.10), and take the transversality conditions in the general form

$$(\lambda_1 \Delta v_r + \lambda_2 \Delta v_\varphi + \lambda_3 \Delta r + \lambda_4 \Delta\varphi - H \Delta t)_{t_i}^{t_f} = 0.$$

Here the total variations of the variables are calculated along the limit orbits. By virtue of formulas (3.10) and (3.11), the variations of all the variables except Δt are expressed in terms of Δf_i and Δf_f. If the formulation of the problem does not involve a specific motion along the limit orbits, the variations Δf_i, Δt_i, Δf_f, and Δt_f are independent. Hence we obtain $H_i = H_f = 0$, and because of the stationarity of equations (3.7) along the whole optimal trajectory $H = 0$. The variations of the velocities and coordinates on the limit orbits are proportional to the right-hand sides of equations (3.7), calculated on the limit orbits for $\dot\beta = 0$.

We obtain the following transversality conditions

$$\sum_{l=1}^{4} \lambda_l^i F_l^i = 0, \quad \sum_{l=1}^{4} \lambda_l^f \; F_l^f = 0, \tag{3.12}$$

where F_l^i and F_l^f are the right-hand sides of equations (3.7) for the initial and final orbits respectively.

On the basis of equations (3.7') we set up the function H for the propulsion periods

$$H = \dot\beta (-1 + \lambda_1 \cos\psi + \lambda_2 \sin\psi). \tag{3.13}$$

The Hamilton-Jacobi equation is

$$\frac{\partial W}{\partial t}+\dot{\beta}\left(-1+\frac{\partial W}{\partial v_r}\cos\psi+\frac{\partial W}{\partial v_\varphi}\sin\psi\right)=0.$$

The complete integral of this equation is

$$W=a_1v_r+a_2v_\varphi+\int_{t_0}^{t}\dot{\beta}\,(1-a_1\cos\psi-a_2\sin\psi)\,dt+a_3r+a_4\varphi.$$

The integrals (3.3) and (3.4) give

$$\begin{gathered}v_r-\int_{t_0}^{t}\dot{\beta}\cos\psi=b_1,\quad v_\varphi-\int_{t_0}^{t}\dot{\beta}\sin\psi=b_2,\quad r=b_3,\\ \varphi=b_4,\quad a_1=\lambda_1,\quad a_2=\lambda_2,\quad a_3=\lambda_3,\quad a_4=\lambda_4.\end{gathered}\tag{3.14}$$

Let us turn our attention to the maximality condition for H on the optimal trajectory. Comparing the first and second partial derivatives of the function (3.13) with respect to $\dot{\beta}$ and ψ, we reach the conclusion that the propulsion periods may correspond to trajectories of: a) programmed thrust if $\partial H/\partial\dot{\beta}=\partial H/\partial\psi=0$ or

$$\lambda_1=\cos\psi,\quad\lambda_2=\sin\psi;\tag{3.15}$$

b) maximum jet acceleration G if $\partial H/\partial\dot{\beta}>0$, $\partial H/\partial\psi=0$. Since $H=0$ and H is linear homogeneous function of $\dot{\beta}$, the inequality sign for the derivative $\partial H/\partial\dot{\beta}$ must be omitted, and the undetermined multipliers are also determined by formulas (3.15).

Formulas (3.14) and (3.15) show that the angle of inclination ψ of the thrust remains constant for any propulsion period. For each such period we obtain, by virtue of (3.7) and (3.14),

$$\begin{gathered}v_r-v_r^0=V\cos\psi,\quad v_\varphi-v_\varphi^0=V\sin\psi,\quad r=r^0,\quad\varphi=\varphi^0,\\ \beta-\beta_0=V,\quad V=[(v_r-v_r^0)^2+(v_\varphi-v_\varphi^0)^2]^{\frac{1}{2}}.\end{gathered}\tag{3.16}$$

Formulas (3.16) imply in particular that the mass flow under impulsive conditions is determined by the required velocity increment and is independent of the specific combustion law.

Let us find the expressions for the undetermined multipliers for the coasting periods by the Hamilton-Jacobi method.

The Hamilton-Jacobi equation has the form

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial v_r}(v_\varphi^2r^{-1}-k^2r^{-2})-\frac{\partial W}{\partial v_\varphi}v_rv_\varphi r^{-1}+\frac{\partial W}{\partial r}v_r+\frac{\partial W}{\partial\varphi}v_\varphi r^{-1}=0.\tag{3.17}$$

The complete integral is

$$W=a_1t+a_2\varphi+\Phi(v_r,\,v_\varphi,\,r,\,a_1,\,a_2,\,a_3,\,a_4),$$

where Φ is some function of the indicated arguments.

On the basis of formula (3.4) we thus have

$$\lambda_4 = \frac{\partial W}{\partial \varphi} = a_2.$$

Therefore, for both coasting and propulsion periods, $\lambda_4 = \mathrm{const}$. By virtue of the Weierstrass-Erdmann conditions, λ_4 is the same constant for the entire optimal trajectory.

For coast trajectories $\dot{\beta} = 0$, when $L = 0$, and on the basis of formula (3.6) the complete integral has the form

$$W(x_i, \lambda_i^0, t) = \sum_{i=1}^{4} x_i^0 \lambda_i^0. \tag{3.18}$$

Using formula (3.4) we obtain expressions for the undetermined multipliers:

$$\lambda_i = \sum_{j=1}^{4} \lambda_j^0 \frac{\partial x_j^0}{\partial x_i}. \tag{3.19}$$

As solutions of equations (3.7) in the coasting periods we take

$$v_r = kep^{-1/2} \sin f,\ v_\varphi = kp^{-1/2}(1 + e\cos f),\ r = p(1 + e\cos f)^{-1},\quad \varphi = f + \omega. \tag{3.20}$$

Here e, p and ω are the eccentricity, a parameter and the angular distance of the pericenter of the elliptical transfer orbit from the polar axis, f is the true anomaly in this orbit.

We note that the true anomaly is expressed in terms of time by means of the relations /8/

$$\mathrm{tg}\,\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\,\mathrm{tg}\,\frac{E}{2},\quad E - e\sin E = ka^{-3/2}(t - t_\pi), \tag{3.21}$$

where t_π is the time of passage through the pericenter, a is the major semiaxis, which is $p(1 - e^2)^{-1}$ for an elliptical orbit.

Denoting by s_k $(k = 1, 2, 3, 4)$ the Keplerian elements e, p, ω and t_π, formula (3.19) assumes the form

$$\lambda_i = \sum_{k=1}^{4} A_k \frac{\partial s_k}{\partial x_i},\quad A_k = \sum_{j=1}^{4} \lambda_j^0 \frac{\partial x_j^0}{\partial s_k} = \mathrm{const}. \tag{3.22}$$

It is immediate that $A_4 = -H$. Introducing the notation $A = -k^{-1}e^{-1}p^{1/2}A_3$, $B = k^{-1}e^{-1}p^{1/2}A_1, D = k^{-1}e^{-1}p^{1/2}(e^2 - 1)A_1 + 2k^{-1}p^{1/2}A_2$, $C = -k^{-1}e^{-1}p^{3}A_4$ and calculating the derivatives $\partial s_k/\partial x_i$ with the aid of relations (3.20) and (3.21), we write expressions (3.22) for the case $e \neq 0$ in the form

$$\begin{gathered}
\lambda_1 = A\cos f + Be\sin f + CI,\\
\lambda_2 = -A\sin f(2 + e\cos f) + B(1 + e\cos f) + D(1 + e\cos f)^{-1} + CJ,\\
\lambda_3 = -kp^{-1/2}(1 + e\cos f)^2[A\sin f(1 + e\cos f)^{-1} - B - D(1 + e\cos f)^{-1} + CK],\\
\lambda_4 = -kep^{-1/2}A.
\end{gathered} \tag{3.23}$$

where I, J and K are Lawden's functions /9/

$$I = \sin f \int \sin^{-2} f (1 + e\cos f)^{-2} df,$$

$$J = e^{-1} c\,\mathrm{tg}\, f (1 + e\cos f)^{-1} + e^{-1} \sin^{-1} f (1 + e\cos f) I,$$

$$\dot{K} = e^{-1} \sin^{-1} f (1 + e\cos f)^{-2} (e \sin^2 f - \cos f) - e^{-1} \operatorname{cosec} f\, I.$$

Formulas of this type for the undetermined multipliers were first obtained by Lawden.

The continuity of the undetermined multipliers, their explicit representations by formulas (3.15) and (3.23), the boundary conditions (3.10) and (3.11), the conditions (3.16) for the velocity increments for each propulsion period and the transversality conditions (3.12) provide sufficient equations to determine all the unknown quantities for a preassigned number of pulses.

§ 4. Optimal impulsive transfer

Investigations by a number of authors have shown that power optimal transfer trajectories between fixed limit orbits may have one, two or three impulsive propulsion periods. Single-impulse transfer is possible if the limit orbits intersect. Three-impulse transfer of the Hohmann type, consisting of twice-tangent ellipses, become optimal if the ratio of the major semi-axes exceeds 11.9. Four-impulse transfer and transfer with a greater number of impulses, based on Hohmann ellipses, cannot be optimal. If the ratio of the semi-axes is less than 11.9, a two-impulse transfer is an optimal Hohmann transfer. Most of the literature deals with two-impulse flights.

The optimality of a two-impulse transfer between coplanar circular orbits along an ellipse tangent to both, was proved by Hohmann in 1925.

In 1959 and 1960 Lawden, Smith, and Long published the results of intense investigations /9—11/, in which they constructed, without calculating specific flights along limit orbits, optimal two-impulse transfers between spatial circular orbits, coplanar elliptical orbits with trimmed axes and a small angle between the lines of apsides, and an optimal two-impulse transfer between noncoplanar orbits with small eccentricities and finite mutual inclination. However, they were pessimistic as to the problem of optimal transfer in the planetary case, in which both the eccentricities of the limit orbits and the mutual inclination are small. The difficulty is that when the eccentricities of the limit orbits and their mutual inclination are made to tend to zero in turn, different limits are obtained.

Using the technique described in /3/ for finding power optimal impulsive transfer, we have succeeded in solving this problem /12/. We start from a general spatial formulation of the problem in spherical coordinates. The result is as follows. If the order of smallness of the inclination is equal to or smaller than that of the eccentricities, the optimal transfer has an ascending node near the ascending node of the final orbit. If the order of smallness of the inclination is greater than that of the eccentricities, the ascending node of the transition orbit is close to the node of the transfer in the case of zero inclination. In both cases the optimal transfer is determined explicitly up to terms of second-order smallness.

This technique has yielded the solution of a number of problems taking into account the specific phases of the motion along the limit orbits, and the internal mass reserve, which is proportional to the total flight time. It was found that even a small internal mass reserve may lead to qualitatively new noncoplanar transfer orbits.

For example, for arbitrary phases of the motion along circular limit orbits (if the ratio of the sum of the characteristic velocities to the effective exhaust velocity is between $\ln 5/3$ and $\ln 3$), the transfer orbit is close to that of transfer without internal reserve; if the ratio is smaller than $\ln 5/3$ or larger than $\ln 3$, the node of the transfer orbit lags 90° behind the node of the orbit obtained without allowing for the internal mass reserve. The inclinations to the initial orbit in these two cases are one order greater than the mutual inclination of the limit orbits. The transfer orbits of the two indicated bases also differ considerably from each other.

Consideration of specific phases of motion along the limit orbits in the case of spatial transfer also leads to similar, qualitatively new, phenomena.

Using the theory of variations we can solve problems with nonimpulsive propulsion phases. We have solved the problem of nonimpulsive short coplanar transfer to a final circular orbit intersecting the initial orbit.

The solution was assumed to be a power series in the angular velocity of the final circular orbit. It turned out that for a solution of first-order accuracy, both programmed schedules and schedules of maximum jet acceleration can be used. Retaining terms of second-order smallness the optimal trajectory is determined uniquely, except for the special case in which the residuals of the transversal and radial velocities are equal.

If the residual in the transversal velocity exceeds that in the radial velocity, the optimal trajectory consists of a coasting phase, followed by a phase of maximum jet acceleration. The maneuver time is found from the optimum conditions and cannot be assigned in advance.

If the residual in the transversal velocity is smaller than that in the radial velocity, the optimal trajectory consists of a propulsion phase with maximum jet acceleration, followed by a coasting phase, and a second propulsion phase with maximum jet acceleration. The maneuver time can be preassigned.

The problem was solved in finite form, giving explicit expressions the times when the engine is switched on. We used Ting-Lu's approximating technique for the optimal trajectory /13/, which we generalized to the case of the variational problem with bounded controls. The solution is described by the author in /14/.

Remark (added in proof). The construction of extremals used in § 2 is not always possible. To prove the necessity of the maximum principle it is possible to select a trajectory AB differing from the optimal trajectory only in some internal time interval including the point (x_i, λ_i, t) and tangent to the field of extremals corresponding to the optimum controls $\{u_k\}$, or intersecting these extremals, at least at two points. The sufficient conditions must be expanded considerably. The conditions indicated by the author are sufficient in the class of trajectories AB for which the required field of extremals can be constructed.

The inaccuracies in our argument regarding the sufficient conditions were pointed out in the discussion of the paper at the conference by A.I. Lur'e, V.V. Rumyantsev, V.A. Troitskii and V.F. Krotov. The inaccuracy of our original results regarding the sufficient conditions was indicated by L. Neustadt (Mathematical Reviews, Vol. 30, No. 2, 1889. 1965).

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V. V. Rumyantsev

INVESTIGATION OF THE STABILITY OF MOTION OF SOLID BODIES WITH LIQUID-FILLED CAVITIES

Problems concerning the motion of solid bodies with cavities partially or completely filled with liquid are largely stimulated by the diverse needs of engineering, but they also are of considerable interest from a theoretical point of view, being on the border between dynamics of solids and hydrodynamics. In recent years a number of studies dealing with these problems have appeared. Nonlinear methods for investigating the stability of motion of such systems, connected mainly with the development of Lyapunov's ideas and methods, have been considerably developed.

The purpose of the three parts of the present paper is to review some of these methods and to indicate their applicability.

In formulating the problem of stability of motion of a solid body with a liquid, which is a system with infinitely many degrees of freedom, one is first faced with the problem of defining the concept of stability: stability in what sense and with respect to what quantities? In his famous thesis /1/, when defining stability for systems with finitely many degrees of freedom, Lyapunov emphasized the necessity of indicating which quantities are assumed to remain small under conditions of stability, since the concept is not absolute.

At present there are three definitions of the stability of motion of liquid-filled solid bodies in the nonlinear formulation.

In the simplest case the motion of the liquid in the cavity is characterized by a finite number of variables. It is known that this is possible when the cavity is completely filled with an ideal liquid, when the motion of the liquid is potential or a uniform rotational motion. In this case it is natural to formulate the stability problem as that of Lyapunov stability for systems with finitely many degrees of freedom /1, 2/.

In general, when no assumptions are made on the motion of the liquid in the cavity other than the natural assumptions of continuity and compactness, the state of the system is described by an infinite number of variables, and the stability problem is much more complicated. However, even in this case it is possible to formulate a problem of stability with respect to a finite number of variables, by considering certain quantities characterizing the motion of the liquid in an integral manner.

Finally, a third definition of stability is connected with Lyapunov's ideas /3, 4/ in the theory of equilibrium figures of a rotating liquid; it is related to the stability of the form of the liquid. In this case the stability problem is reduced to minimizing a certain expression, in general a functional.

We dealt with the first two approaches to the stability problem in our review /9/, and therefore in the present paper we consider them as briefly as possible.

I. COMPLETELY FILLED CAVITY; SIMPLEST CASES

1. Irrotational motion

Consider a solid body with a simply- or multiply-connected cavity of arbitrary form, completely filled with an ideal homogeneous incompressible liquid subject to potential mass forces. Suppose the motion of the liquid is irrotational, the velocity vector $\mathbf{v} = \operatorname{grad} \Phi$. Due to the incompressibility of the liquid, the velocity potential Φ is a harmonic function of the coordinates, whose derivative in the direction of the normal at the walls of the cavity is equal to the normal component of the velocity of the corresponding point of the wall, and the circulations along the principal contours inside a multiply-connected cavity are given constants k_j determined by the motion of the liquid in the solid body at rest.

This case was studied thoroughly by N. E. Zhukovskii /5/, who established a number of fundamental theorems. Following Zhukovskii, the function is determined by decomposing the motion of the solid body into a translation with the velocity of some point Φ and a rotation about this point. The translational motion has a unique potential function, so that it is communicated directly to the liquid and does not influence its internal motion, which varies only because of the rotation of the body about the point O. The velocity potential, expressed in a moving system of coordinate axes $Oxyz$ rigidly attached to the body, is assumed to have the form

$$\Phi(x, y, z, t) = w_1 x + w_2 y + w_3 z + \sum_{i=1}^{3} \omega_i \psi_i(x, y, z) + \sum_{j=1}^{r-1} k_j \vartheta_j(x, y, z), \qquad (1.1)$$

where w_i and ω_i are the projections of the velocity vector of the point O and of the instantaneous angular velocity vector of the body, $\psi_i(x, y, z)$ and $\vartheta_j(x, y, z)$ are harmonic functions which satisfy at the walls σ of the cavity the conditions:

$$\frac{d\psi_1}{dn} = yn - zm, \quad \frac{d\psi_2}{dn} = zl - xn, \quad \frac{d\psi_3}{dn} = xm - yl,$$
$$\frac{d\vartheta_j}{dn} = 0 \quad (j = 1, \ldots, r-1), \qquad (1.2)$$

where the function $\vartheta_j(x, y, z)$ decreases by unity when passing through the j-th partition in the direction of the j-th principal contour and varies continuously in passing through the remaining partitions, which can be constructed in the cavity in order to make it simply-connected /6/; the components of the unit normal vector $\mathbf{n}$ are denoted by l, m, n.

The motion of the liquid in the cavity is thus determined by a finite number of quantities w_i, ω_i $(i = 1, 2, 3)$ and k_j $(j = 1, \ldots, r-1)$, and the motion of the whole system (body with liquid) is described by the equations of the dynamics of solid bodies together with the Laplace equation.

The mechanical effect of a liquid which has no initial velocities in the body at rest is identical to the effect of some equivalent body attached to the given solid body. If the liquid in a multiply-connected cavity has some initial irrotational motion, it produces an additional effect similar to that of some revolving rotor attached to the solid body. This important theorem is due to Zhukovskii /5/.

Thus, in the case of irrotational motion of an ideal liquid completely filling the cavity of a body, the motion of the body is determined by a finite number of variables. In this case, therefore, it is natural to formulate the stability problem in the sense of Lyapunov stability for systems with finitely many degrees of freedom. N.G. Chetaev formulated and solved this problem for a number of cases /7/.

Example 1 /10/. Let a solid body with liquid move about a stationary point O under the action of forces with a force function $U(\gamma_3)$ /8/, γ_i $(i = 1, 2, 3)$ denoting the direction cosines of the stationary axis $O\zeta$ relative to the moving axes x, y, z. We assume that the initial motion of the liquid in the multiply-connected cavity is such that the only nonzero component of the angular momentum of the liquid is that along the z axis, R. Under these conditions the equations of motion have a particular solution which describes uniform rotation of the solid body about the ζ axis with angular velocity ω, provided that $(\partial U/\partial\gamma_3)_{\gamma_3=1} \neq \infty$.

Let us study the stability of this motion with respect to ω_i, γ_i $(i = 1, 2, 3)$. First consider the case of a dynamically symmetric body, when $A = B$. Using the known first integrals of the equations of perturbed motion /1/, we follow Chetaev's method /2/ and construct the function

$$V = A(\omega_1^2 + \omega_2^2) + 2A\lambda(\omega_1\gamma_1 + \omega_2\gamma_2) + 2C\lambda\xi\eta -$$
$$- \left[C\omega\lambda - \left(\frac{\partial U}{\partial\gamma_3}\right)_1 + R\lambda\right](\gamma_1^2 + \gamma_2^2 + \eta^2) + \frac{C^2}{A}\xi^2 + \ldots, \tag{1.3}$$

where λ is a constant; the time-derivative of the function V vanishes by virtue of the equations of perturbed motion. Since by the condition $V' = 0$.

$$(C\omega + R)^2 + 4A\left(\frac{\partial U}{\partial\gamma_3}\right)_1 > 0, \tag{1.4}$$

the function V is positive definite, Lyapunov's theorem implies that inequality (1.4) is a sufficient condition for stability of the unperturbed motion.

Considering the function $W = \omega_1\gamma_2 - \omega_2\gamma_1$, we easily see, on the basis of Chetaev's instability theorem, that (1.4) is also a necessary condition for stability.

In the case $A > B$, a sufficient condition for stability can be obtained in a similar way:

$$(C - A)\omega^2 + R\omega + \left(\frac{\partial U}{\partial\gamma_3}\right)_1 > 0. \tag{1.5}$$

2. Homogeneous rotational motion of the liquid /11/

Suppose a homogeneous incompressible ideal liquid completely fills an ellipsoidal cavity:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1. \tag{1.6}$$

The velocity of the liquid can be represented in the form

$$\mathbf{v} = \operatorname{grad}\varphi + \boldsymbol{\Omega} \times \mathbf{r}, \tag{1.7}$$

where the harmonic function

$$\begin{aligned} \varphi(x, y, z, t) = {} & w_1 x + w_2 y + w_3 z + \\ & + [(b^2 - c^2) yz + 2(c^2 y_0 z - b^2 z_0 y)] \frac{\omega_1 - \Omega_1}{b^2 + c^2} + \\ & + [(c^2 - a^2) xz + 2(a^2 z_0 x - c^2 x_0 z)] \frac{\omega_2 - \Omega_2}{c^2 + a^2} + \\ & + [(a^2 - b^2) xy + 2(b^2 x_0 y - a^2 y_0 x)] \frac{\omega_3 - \Omega_3}{a^2 + b^2} \end{aligned} \tag{1.8}$$

satisfies the boundary conditions at the walls of the cavity, and the vector $\boldsymbol{\Omega}(t)$ satisfies the Helmholtz equation

$$\frac{d\boldsymbol{\Omega}}{dt} + \boldsymbol{\omega} \times \boldsymbol{\Omega} = (\boldsymbol{\Omega} \cdot \nabla)\mathbf{v}. \tag{1.9}$$

Thus, in this case also the motion of the solid body with the liquid is completely characterized by a finite number of variables, which makes it possible to formulate the problem of Lyapunov stability for systems with finitely many degrees of freedom.

For examples, see /9, 10, 11, 12/.

III. STABILITY OF MOTION OF A LIQUID-FILLED SOLID BODY WITH RESPECT TO SOME OF THE VARIABLES

1. On the equations of motion

Imagine a free or constrained solid body with a cavity partially or completely filled with an ideal or viscous liquid. Considering the body and the liquid as a single mechanical system subject to given forces, it is easy to set up its equations of motion in some form; in general we get a system of ordinary and partial differential equations with appropriate initial and boundary conditions /13/.

For example, if the solid body has a fixed point, the equations of motion can be written

$$\begin{gathered} \frac{d(\mathcal{G}_1 + \mathcal{G}_2)}{dt} + \boldsymbol{\omega} \times (\mathcal{G}_1 + \mathcal{G}_2) = \mathbf{L}, \\ \frac{d\mathbf{v}}{dt} + \boldsymbol{\omega} \times \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \operatorname{grad} p + \nu \Delta \mathbf{v}, \\ \operatorname{div} \mathbf{v} = 0, \end{gathered} \tag{2.1}$$

where $\mathcal{G}_i$ is the momentum of the body ($i = 1$) and of the liquid ($i = 2$), $\mathbf{L}$ is the moment of the external forces applied to the system, $\mathbf{F}$ is the vector of mass forces acting on the liquid, ρ is the density, $p(x, y, z, t)$ is the pressure, Δ is the Laplace operator, μ is the coefficient of viscosity, $\nu = \mu/\rho$.

The velocity vector of the liquid is $\mathbf{v} = \mathbf{v}_e + \mathbf{u}$, where $\mathbf{u}$ is the velocity vector of the liquid relative to the body, $\mathbf{v}_e$ is the translational velocity vector.

The boundary conditions are: at the cavity walls σ_1 with which the viscous liquid is in contact:

$$\mathbf{u} = 0, \tag{2.2}$$

at the free surface σ_f, whose equation we suppose to be $f(x, y, z, t) = 0$, we have the kinematic condition

$$\partial f/\partial t + \mathbf{u}\cdot \operatorname{grad} f = 0 \tag{2.3}$$

and the dynamic condition

$$\mathbf{p}_n = -p_0\mathbf{n}. \tag{2.4}$$

Here $\mathbf{p}_n$ denotes the stress vector, $\mathbf{n}$ is the unit vector on the external normal, p_0 is the constant air pressure over the free surface. In the case of an ideal liquid ($\nu = 0$), conditions (2.2) and (2.4) are replaced by

$$u_n = 0 \text{ at } \sigma_1, \quad p = p_0 \text{ at } \sigma_f, \tag{2.5}$$

The equations of a solid body with liquid admit, under certain conditions, an integral relation and first integrals which may prove useful in stability investigations.

Thus, if the forces acting on the system have a force function U, and the constraints imposed on the body are ideal and stationary, the following equation is easily obtained:

$$\begin{aligned}\frac{d}{dt}(T-U) = -\mu \int\limits_{\tau}\Big\{2\Big[\Big(\frac{\partial v_1}{\partial x}\Big)^2 + \Big(\frac{\partial v_2}{\partial y}\Big)^2 + \Big(\frac{\partial v_3}{\partial z}\Big)^2\Big] + \\ + \Big(\frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial z}\Big)^2 + \Big(\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x}\Big)^2 + \Big(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\Big)^2\Big\}\, d\tau,\end{aligned} \tag{2.6}$$

where τ is the volume of the liquid.

Hence it follows that motion of a viscous liquid without dissipation of energy is possible only if in the whole liquid

$$\frac{\partial v_1}{\partial x} = \frac{\partial v_2}{\partial y} = \frac{\partial v_3}{\partial z} = 0, \ \frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial z} = \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} = \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{2.7}$$

i.e., if the liquid and the body move as a single rigid body. For an ideal liquid, equation (2.6) yields the energy integral

$$T - U = h. \tag{2.8}$$

If the moment of the external forces acting on the system is zero relative to some stationary axis, say the ζ-axis, then the component of the angular momentum $\mathcal{G}_\zeta$ of the system along this axis remains constant during the entire motion.

If the ellipsoid of inertia of the body and the cavity is a solid of revolution, say about the z-axis, and the moment of the external forces with respect to this axis is zero, then in the case of an ideal liquid the component of the instantaneous angular velocity of the body along this axis, ω_3, remains constant /13/.

2. Stability with respect to some of the variables

The position of the solid body and the liquid relative to a stationary system of coordinates axes $O_1\xi\eta\zeta$ can be defined by the Lagrangian coordinates of the body q_j ($j = 1, \ldots n \leqslant 6$) and absolute ξ, η, ζ or relative x, y, z coordinates of the liquid particles, which are infinite in number.

Suppose the equations of motion have some particular solution

$$\begin{gathered} q_j = f_j(t), \quad q'_j = f'_j(t) \quad (j = 1, \ldots, n), \\ v_i = F_i(x, y, z, t) \quad (i = 1, 2, 3), \\ p = F_4(x, y, z, t), \end{gathered} \tag{2.9}$$

which satisfies certain initial conditions

$$\begin{gathered} q_{j0} = f_j(t_0), \quad q'_{j0} = f'_j(t_0), \\ v_i(x, y, z, t_0) = \varphi_i(x, y, z) \end{gathered} \tag{2.10}$$

(where t_0 is the initial instant), and also boundary conditions.

The solution (2.9) describes some motion of the solid body with the liquid; let us take this as the unperturbed motion and investigate its stability. Suppose that for the perturbed motion we have, instead of (2.10), initial conditions

$$\begin{gathered} q_{j0} = f_j(t_0) + \varepsilon_j, \quad q'_{j0} = f'_j(t_0) + \varepsilon'_j, \\ v_i = \varphi_i(x, y, z) + \varepsilon_i(x, y, z), \end{gathered} \tag{2.11}$$

where ε_j and ε'_j are real constants, while $\varepsilon_i(x, y, z)$ are real functions satisfying the equation of continuity, which we shall call perturbations. We assume the perturbations to be sufficiently small in absolute value.

We obtain the equations of perturbed motion from the original equations of motion of the system by substituting

$$\begin{gathered} q_j = f_j(t) + x_j, \quad q'_j = f'_j(t) + x'_j, \\ v_i = F_i(x, y, z, t) + u_i(x, y, z, t), \quad p = F_4(x, y, z, t) + p^*(x, y, z, t). \end{gathered} \tag{2.12}$$

If it turns out that for any perturbations ε_j, ε'_j, $\varepsilon_i(x, y, z)$ of sufficiently small absolute value the deviations or variations of x_j, x'_j ($j = 1, \ldots, n$), u_i ($i = 1, 2, 3$), p^* are arbitrarily small in absolute value for any $t > t_0$, then the unperturbed motion (2.9) is stable with respect to q_j, q'_j, v_i and p.

However, in this formulation the problem presents tremendous difficulties, and at present the only known methods for its solution are those of the theory of small oscillations.

These difficulties may be avoided by varying the formulation of the problem. The point is that in applied problems of stability of motion of solid bodies with liquid-filled cavities we are interested mainly in the stability of motion of the solid body; the stability of motion of the liquid in the cavity is of interest only inasmuch as it affects the stability of motion of the solid body. Of course, these two aspects of the single stability problem are interrelated: the motion of the body depends on that of the liquid, and vice versa.

It is therefore natural to formulate the problem in the Lyapunov sense with respect to the variables which determine the motion of the solid body, and to some quantities

$$p_s = \int_\tau \Phi_s(x, y, z, v_1, v_2, v_3)\, d\tau,$$

which characterize the motion of the liquid in an integral manner. Here $\Phi_s(x, y, z, v_1, v_2, v_3)$ are certain real, continuous, bounded functions. Thus, for example, it follows from equations (2.1) that for given external forces acting on the system, the motion of the solid body depends on the angular momentum of the liquid and its rate of variation with time, which in turn depend on the motion of the body. In this problem, therefore, an appropriate choice for the p_s is the components of the angular momentum of the liquid.

We note that the p_s—the momentum, angular momentum of the liquid, etc.—do not characterize the motion of the liquid (which depends on an infinite number of variables) completely, but only in an integral manner. This being so, the stability of motion of the liquid with respect to the p_s, is conditional stability—the stability of motion of a continuous medium with respect to some of the infinitely many variables characterizing its motion.

In this approach, our problem, which refers to a system with infinitely many degrees of freedom, is reduced to the investigation of stability with respect to a finite number of quantities q_j, q_j', p_s, and this ensures stability of motion of the solid body with respect to q_j, q_j'.

Using this formulation of the problem /14/, the methods of the theory of stability of systems with finitely many degrees of freedom, in particular, Lyapunov's second method, may be applied to its solution.

Consider some given continuous real functions Q_r of the quantities q_j, q_j', p_s and the time t. For the unperturbed motion, the functions Q_r, after substituting (2.9), are transformed into certain known functions of time $F_r(t)$, and for the perturbed motion (2.12) (maintaining the previous notation Q_r) they are functions of the time and the variables x_j, x_j' and u_l, which depend in turn on the perturbations $\varepsilon_j, \varepsilon_j', \varepsilon_l(x,y,z)$. Consider the differences $y_r = Q_r - F_r$, which depend on the time t and the variables x_j, x_j' and u_l. Let L_r be arbitrary positive numbers. If for any L_r, however small, there exist positive numbers E_j, E_j', E_l^* such that for any perturbations $\varepsilon_j, \varepsilon_j', \varepsilon_l(x, y, z)$ satisfying the conditions $|\varepsilon_j| \leqslant E_j$, $|\varepsilon_j'| \leqslant E_j'$, $|\varepsilon_l(x, y, z)| \leqslant E_l^*$, and for any $t > t_0$ we have $|Q_r - F_r| < L_r$, then the unperturbed motion is stable with respect to the quantities Q_r, and unstable otherwise /1/.

We present a theorem /15/ which is useful in some cases as a modification of Lyapunov's stability theorem.

Suppose there exists a real, continuous, bounded, single-valued function $\varphi(t, x_j, x_j', u_l)$ which vanishes for $x_j = x_j' = u_l = 0$, whose time-derivative $\varphi' \leqslant 0$ by virtue of the equations of perturbed motion. Suppose that for all relevant values of the variables t, x_i, x_i', u_i and the corresponding values of the variables y_r we have the inequality

$$\Psi(t, y_1, \ldots, y_k) \leqslant \varphi(t, x_i, x_i', u_i), \tag{2.13}$$

where $\Psi(t, y_r)$ is a real, continuous, bounded, single-valued function which vanishes when all $y_r = 0$ $(r = 1, \ldots, k)$.

Theorem 2.1. *If the differential equations of perturbed motion are such that there exists a function* $\varphi(t, x_i, x_i', u_i)$ *whose time derivative* $\varphi' \leqslant 0$, *and a positive definite function* $\Psi(t, y_1, \ldots, y_k)$ *such that inequality (2.13) holds, then the unperturbed motion is stable with respect to* Q_r $(r = 1, \ldots, k)$.

Example 2. Consider a solid body with a cavity completely filled with a viscous liquid, moving about a fixed point O due to forces with a force function $U(\gamma_3)$ /16/. Included among the real motions are uniform rotations of the entire system as a single rigid body about a stationary axis $O\zeta$. The problem of stability of such motions with respect to the quantities ω_i, ϑ_{2i}, γ_i $(i = 1, 2, 3)$ can be solved by considering the function

$$\begin{aligned} V = {} & A_1\omega_1^2 + B_1\omega_2^2 + C_1\xi^2 - 2\omega\,[(A_1\omega_1 + \vartheta_{21})\gamma_1 + \\ & + (B_1\omega_2 + \vartheta_{22})\gamma_2 + (C_1\xi + \eta)\zeta] + \frac{1}{A_2}\vartheta_{21}^2 + \frac{1}{B_2}\vartheta_{22}^2 + \\ & + \frac{1}{C_2}\eta^2 + \left(C\omega^2 + \left(\frac{\partial U}{\partial\gamma_3}\right)_1\right)(\gamma_1^2 + \gamma_2^2 + \zeta^2) + \left(\lambda - \left(\frac{\partial^2 U}{\partial\gamma_3^2}\right)_1\right)\zeta^2, \end{aligned} \tag{2.14}$$

where the constant $\lambda > (\partial^2 U/\partial\gamma_3^2)_1 - (\partial U/\partial\gamma_3)_1$. For the function V to be positive definite, it is necessary and sufficient that the following inequality be satisfied:

$$(C - A)\,\omega^2 + (\partial U/\partial\gamma_3)_1 > 0 \quad (A > B). \tag{2.15}$$

It is easily seen /10/ that the conditions of Theorem (2.1) are satisfied, so that inequality (2.15) is a sufficient condition for the stability of the unperturbed motion.

In a similar formulation, N. N. Kolesnikov /17/ solved the problem of the stability of motion of a free solid body with a cavity filled with a viscous liquid in a Newtonian central force field. He proved that the conditions

$$C > A > B \tag{2.16}$$

are sufficient for the motion of the body along a circular orbit to be stable. Conditions (2.16) have the same form as those established by V. V. Beletskii /18/ for the case of a solid body alone.

III. STABILITY OF STEADY-STATE MOTIONS OF A LIQUID-FILLED SOLID BODY

1. Some formulas

Consider a solid body having a simply-connected cavity partially or completely filled with a homogeneous incompressible ideal liquid. Suppose that stationary ideal constraints are imposed on the body, given forces applied to the body, and the mass forces acting on the liquid particles have force functions U_1 and U_2 which do not depend explicitly on the time. As before, we shall assume that the motions of both the body and the liquid are continuous, so that the coordinates of each particle of the liquid are continuous functions of their initial values and time.

Under these conditions the differential equations of motion of the system have an energy integral

$$T + V = h, \tag{3.1}$$

where V is the potential energy of the forces acting on the system, and h is an integration constant.

The position of the solid body relative to a system of coordinates $O_1\xi\eta\zeta$ is defined by its Lagrangian coordinates $q_1, \ldots, q_n$ $(n \leqslant 6)$. The potential energy V of the system is in general a function of the coordinates $q_1, \ldots, q_r$ $(r \leqslant n)$ and the form of the liquid inside the cavity.

We also assume that the constraints imposed on the body permit rotation of the whole system as a single rigid body about some fixed axis, about which the forces acting on the system have no moment. Then the potential energy V of the system obviously does not depend on the angle of rotation q_n of the body about the fixed axis. Under these conditions there exists an area integral for the plane orthogonal to this axis. Taking this axis as the axis $O_1\zeta$ of the stationary coordinate system, the area integral has the form

$$\mathcal{G}_\zeta = k. \tag{3.2}$$

Together with the stationary coordinate system we introduce a system of coordinate axes $O_1\xi_1\eta_1\zeta$ rotating about the ζ-axis with some angular velocity ω. We denote by $\mathbf{v}$ (v_1, v_2, v_3) the absolute velocity vector of any given point of the body or liquid with radius-vector $\mathbf{r}$ (ξ, η, ζ), and by $\mathbf{u}$ (u, v, w) the velocity vector of the same point relative to the system $O_1\xi_1\eta_1\zeta$; then $\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}$.

The kinetic energy and the ζ-component of the angular momentum of the system can then be represented in the form

$$T = T_r + \omega\mathcal{G}^1_\zeta + \frac{1}{2}\omega^2 S, \quad \mathcal{G}_\zeta = \mathcal{G}^1_\zeta + \omega S, \tag{3.3}$$

where

$$T_r = \frac{1}{2}\sum_\nu m_\nu(u_\nu^2 + v_\nu^2 + w_\nu^2), \quad \mathcal{G}^1_\zeta = \sum_\nu m_\nu(\xi_\nu v_\nu - \eta_\nu u_\nu),$$

$$S = \sum_\nu m_\nu(\xi_\nu^2 + \eta_\nu^2).$$

The angular velocity ω of the rotating system $O_1\xi_1\eta_1\zeta$ may be assigned arbitrarily; we agree to choose it /4/ so that at any instant the ζ-component of the relative angular momentum of the system is zero $\mathcal{G}^1_\zeta = 0$, which, by virtue of (3.2) and (3.3), is equivalent to the equation

$$\omega S = k. \tag{3.4}$$

With this choice of ω, the energy integral (3.1) can be represented in the form

$$T_* + \frac{1}{2}\frac{k^2}{S} + V = h. \tag{3.5}$$

Under the above assumptions regarding the constraints and the forces, the system may indeed perform uniform rotations as a single rigid body about the stationary axis ζ. In this case the system is in equilibrium with respect to the system $O_1\xi_1\eta_1\zeta$, which rotates about the ζ axis at uniform angular velocity ω_0.

The general equation of dynamics then leads to the equality

$$\delta U = 0, \quad U = \frac{1}{2}\omega_0^2 S - V, \tag{3.6}$$

where the symbol δ denotes variation corresponding to virtual displacements of the system which are compatible with the constraints. But the expression for U may be regarded as the force function of the given forces and the centrifugal inertial forces. By the principle of virtual displacements, equality (3.6) represents the condition of equilibrium of the system with respect to the coordinate axes $O_1\xi_1\eta_1\zeta$, if the latter rotate at constant angular velocity ω_0.

Let us introduce the expression

$$W = \frac{1}{2}\frac{k_0^2}{S} + V, \tag{3.7}$$

where k_0 is the value of the area constant k for uniform rotation of the whole system as a single rigid body about the ζ-axis at angular velocity ω_0. The variation of this expression corresponding to virtual displacements of the system is $\delta W = -\frac{1}{2}\frac{k_0^2}{S_0^2}\delta S + \delta V$, where S_0 is the value of S for steady-state motion.

Comparing δW with δU and using the fact that $\omega_0 S_0 = k_0$, we see that (3.6) is equivalent to the equation

$$\delta W = 0. \tag{3.8}$$

Thus, in the case of steady-state motion of the system, expression (3.7) has an extremal (stationary) value.

According to its definition (3.7), W depends on the same coordinates of the body $q_1, \ldots, q_{n-1}$ as S and V, on the form of the liquid and on the value of the constant k_0.

Condition (3.8) leads, as is easily seen, to the equations

$$\frac{\partial W}{\partial q_i} = -\frac{1}{2}\omega_0^2\frac{\partial S}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{3.9}$$

for the coordinates q_i of the solid body in steady-state motion, and also to equations for the pressure in the liquid, which yield the equation

$$\frac{1}{2}\omega_0^2(\xi^2 + \eta^2) + U_2(\xi, \eta, \zeta) = \text{const} \tag{3.10}$$

of its free surface in this motion, if the liquid does not fill the cavity completely.

For a fixed value of the parameter k_0, equations (3.9) and (3.10) determine the coordinates of the solid body and the form of the free surface of the liquid in steady-state motion.

2. Stability theorem

Consider some steady-state motion of the system corresponding to a given value of the area constant k_0. Without loss of generality, let us assume that the roots of equations (3.9) for the given value of k_0 are $q_i = 0\ (i = 1, \ldots, n-1)$. In this case the liquid has a form of relative equilibrium f_0, bounded by the free surface σ_0 defined by equation (3.10) and the cavity walls with which the liquid is in contact.

Let us investigate the stability of this steady-state motion using the energy integral (3.5).

If the liquid completely fills the cavity, stability of motion will mean stability in the Lyapunov sense /1/, with respect to the noncyclic coordinates $q_1, \ldots, q_{n-1}$ of the body on which the potential energy V and the moment of inertia S of the system depend explicitly, the generalized velocities $q_1', \ldots, q_n'$ and the kinetic energy $T^{(2)}$ of the liquid.

If the cavity is only partially full and the liquid has a free surface, the case is more complicated. As shown by Lyapunov, /3/, the energy integral does not ensure, in general, that the liquid, after its state of rest (absolute or relative) has been disturbed, have the type of motion which serves as a criterion for stable equilibrium in the mechanics of systems with finitely many degrees of freedom. Lyapunov established that the difficulty disappears only if the stable equilibrium form is defined in such a way that, after imparting to the liquid sufficiently small perturbations, its form remains only slightly different from this equilibrium form, unless arbitrarily thin string or leaf projections form on the surface of the liquid.

We adopt this definition and, following Lyapunov, introduce some of the relevant concepts as applied to our problem. We shall compare the relative equilibrium form of the liquid f_0 with its form f at any instant of the perturbed motion, disregarding the motion of the liquid particles themselves but taking the kinetic energy of the liquid into account. The form f is bounded by the free surface σ_f of the liquid and the cavity walls with which the liquid is in contact at the given instant. For perturbed motion sufficiently close to the unperturbed motion, in a system of coordinates xyz rigidly attached to the body, the forms f_0 and f differ only in the free surfaces σ_0 and σ_f. Owing to the incompressibility of the liquid, the volume of the form f is of course equal to that of the form f_0.

Consider some point P of the surface σ_f and the point P_0 nearest to it on the surface σ_0. As the position of the point P varies on the surface σ_f the distance PP_0 also varies, and for some position of the point P becomes the greatest possible distance for the given instant. Lyapunov called this maximum distance PP_0 the separation; let us denote it by l. We also introduce the deviation ∇ of the liquid form f from the equilibrium form f_0, defining it as the volume of the part of the form f outside the form f_0, or, equivalently, the volume of the part of f_0 outside f.

It is obvious that for a given separation l, the deviation ∇ has a certain maximum of the form $l\psi(l)$, where $\psi(l)$ is a positive function none of whose values exceed a certain limit. For l not exceeding some given number A, the function $\psi(l)$ has a nonzero minimum. The minimum deviation ∇ for a given value of l is always zero /4/.

For continuous motion of the body and liquid the separation l and the deviation ∇ are obviously continuous functions of time.

If for any arbitrarily small positive numbers L_1 and L_2, there exists a positive number λ such that for any initial values of the coordinates q_{i0} and the generalized velocities q'_{i0} $(i = 1, \ldots, n-1)$ of the body, the separation l_0, the deviation ∇_0 and the relative velocities of the liquid u_0, v_0, w_0, satisfying conditions

$$|q_{i0}| \leqslant \lambda, \quad |q'_{i0}| \leqslant \lambda, \quad |l_0| \leqslant \lambda, \quad \nabla_0 > \varepsilon l_0,$$
$$|u_0| \leqslant \lambda, \quad |v_0| \leqslant \lambda, \quad |w_0| \leqslant \lambda, \tag{3.11}$$

and for any $t > t_0$ or, at least, for

$$\nabla > \varepsilon l \tag{3.12}$$

we have inequalities

$$|q_i| < L_1, \quad |l| < L_1, \quad |q'_i| < L_2, \quad |T_{\bullet}^{(2)}| < L_2, \tag{3.13}$$

then the unperturbed steady-state motion of the solid body with liquid is called stable; otherwise unstable.

Here ε denotes a positive number smaller than the minimum of the function $\psi(l)$ under the condition $|l| \leqslant L_1$, and the quantity εl can be regarded as the possible deviation of the liquid. Note that if the liquid completely fills the cavity, the condition connected with (3.12) is omitted.

We also need the concept of the minimum for W. If W is a function $W(q_1, \ldots, q_{n-1})$, its minimum for a fixed value of the constant k_0 is an isolated minimum with respect to those variables $q_1, \ldots\ldots, q_{n-1}$ on which it explicitly depends. If the liquid does not fill the cavity completely, then, following Lyapunov /3/, we use the following definition of the minimum for W.

If for the steady-state motion in question ($q_i = 0$ $(i = 1, \ldots, n-1)$, $l = 0$, $\nabla = 0$) W_0 is the minimum of W, then there exists a sufficiently small positive number E such that for all values of the coordinates q_i $(i = 1, \ldots, n-1)$, the separation l and the deviation ∇, such that $|q_i| \leqslant E, |l| \leqslant E$, $\nabla > \varepsilon l$, where ε is a positive number smaller than the minimum of the function $\psi(l)$ under the condition $|l| \leqslant E$, the difference $W - W_0$ is always positive, vanishing only for $q_i = 0$, $l = 0$, $\nabla = 0$.

We note that for any given value of l, the difference $W - W_0$ may be made arbitrarily small by appropriate selection of a position of the body and a form of the liquid for which $|q_i|$ and ∇ are sufficiently small. But the limit case, in which for $l \neq 0$ all $q_i = 0$ $(i = 1, \ldots, n-1)$, ∇, and consequently also $W - W_0$ vanish is obviously impossible, if only forms which the liquid can assume are considered. The condition $\nabla > \varepsilon l$ was introduced to eliminate this difficulty.

Theorem 3.1. *If $W = \frac{1}{2}\frac{k_0^2}{S} + V$ has an isolated minimum W_0 for the steady-state motion of a solid body with a liquid-filled cavity, the unperturbed motion is stable.*

The simple proof of this theorem /19/ is based on the use of the energy integral (3.5), which we write in the form

$$T_* + W + \frac{1}{2}\frac{k^2 - k_0^2}{S} = T_*^{(0)} + W^{(0)} + \frac{1}{2}\frac{k^2 - k_0^2}{S^{(0)}}, \tag{3.14}$$

where (0) denotes the initial value of the respective quantity; k is the value of the area constant for the perturbed motion.

Remark 1. Lyapunov /4/ pointed out that in order to characterize the difference between the perturbed and the unperturbed form of the liquid it is possible to replace the separation l by some other quantities which may vanish only for the unperturbed form. One may, for example, take the deviation ∇, and prove as before that if, for the considered steady-state motion of a solid body with a liquid partially filling the cavity, W has a minimum W_0 in the sense that $W - W_0 > 0$ for all values of $|q_i|$ and ∇ not simultaneously zero and smaller than a fixed constant, then (for sufficiently small perturbations) $q_i, \dot{q}_i$ $(i = 1, \ldots, n-1)$, ∇ and $T_*^{(0)}$ remain smaller than any preassigned, arbitrarily small, constants during the entire motion.

Corollary. If the potential energy V of the system has an isolated minimum V_0 for the equilibrium position of the solid body with a liquid-filled cavity, (for $k_0 = 0$) the equilibrium position is stable /20/. We note that this result is also applicable to the case of relative equilibrium of a solid body with a liquid-filled cavity.

Suppose that for some fixed value of the parameter W has a minimum, i.e., the steady-state motion is stable. Let us now vary the parameter k_0 continuously, so that the roots of equations (3.9) describe some branch C of the "equilibrium" curve. The expression for W will then vary continuously; for all points of the curve C for which W remains a minimum, the steady-state motions are stable. A change of stability on this branch may occur only at bifurcation points.

The above theorems remain valid for a viscous liquid. For example, in the proof of Theorem 3.1 we need only replace equation (3.14) by the inequality

$$T_* + W + \frac{1}{2}\frac{k^2 - k_0^2}{S} < T_*^{(0)} + W^{(0)} + \frac{1}{2}\frac{k^2 - k_0^2}{S^{(0)}}, \tag{3.15}$$

changing nothing in the proof. Moreover, under the conditions of Theorem 3.1 it can be proved that any perturbed motion sufficiently close to the unperturbed motion tends in the limit to the steady-state motion of the system as a single rigid body. If, however, for an isolated steady-state motion of a solid body with a cavity filled with a viscous liquid W has no minimum, this motion is unstable.

Hitherto we have not taken the surface tension of the liquid into consideration. In a number of cases, however, particularly in conditions of weightlessness of "low gravity", this factor may be important.

The theorems proved in this section remain valid also when the surface tension of the liquid is taken into account, if the potential energy of the surface tension forces $\alpha\sigma + \alpha_1\sigma_1 + \alpha_2\sigma_2$ is added to the criterion function of the stability problem. Here $\alpha, \alpha_1, \alpha_2$ denote the coefficients of surface tension, $\sigma, \sigma_1, \sigma_2$ are the areas of the free surface of the liquid and the surfaces of contact of the liquid and air with the walls of the cavity. Then, for example, a theorem analogous to 3.1 may be formulated as follows.

If the expression

$$W = \frac{1}{2}\frac{k_0^2}{S} + V + \alpha\sigma + \alpha_1\sigma_1 + \alpha_2\sigma_2 \tag{3.16}$$

has an isolated minimum W_0 for steady-state motion of a solid body with a liquid-filled cavity, the unperturbed motion is stable.

The proof is completely analogous to that of Theorem 3.1.

3. The minimum problem

We now proceed to the solution of the minimum problem. Since the minimum property is local in the sense that values of the W at a given point are compared with its values at all points of a sufficiently small neighborhood, a rigorous solution of the minimum problem, excluding some special cases, may be confined to consideration of second-order quantities. The methods of the theory of small oscillations are thus very convenient, if the displacement l of the free surface from its equilibrium position is represented as a series in a system of eigenfunctions of the corresponding boundary-value problem. This method was used, for example, by N. N. Moiseev /21/ and G.S. Narimanov /22/ to solve the problem of the minimum potential energy of a heavy pendulum containing a cavity filled with a heavy liquid, and by V.V. Rumyantsev /19/ to solve the problem of the minimum of W for a heavy solid body containing a cavity filled with a weightless liquid.

However, the calculations are very tedious since one must operate with infinite series, multiply them, integrate them over certain regions, etc.

In this respect, another method, the main idea of which is due to G.K. Pozharitskii /23/, is more convenient. He pointed out that the potential energy of a heavy liquid has a minimum when the container deviates from the equilibrium position, if the free surface of the liquid is a horizontal plane.

We describe the solution of the minimum problem for W /24/.

In the neighborhood of the steady-state motion of a solid body with a liquid, for which $q_i = 0$ $(i = 1, \ldots, n-1)$, consider a region of the variables

$$|q_i| < H, \tag{3.17}$$

where $H > 0$ is a sufficiently small constant. Let q_i $(i = 1, \ldots\ldots, n-1)$ be the coordinates of some fixed points of the region (3.17). Let us find the form of the free surface of the liquid such that W has an extremum for the given fixed q_i. To solve this problem, we find the first variation of W for fixed q_i and equate it to zero:

$$\delta W = -\rho\int_\tau \left[\frac{1}{2}\frac{k_0^2}{S^2}\delta(\xi^2 + \eta^2) + \delta U_2(\xi, \eta, \zeta)\right] d\tau = 0, \tag{3.18}$$

where S is the moment of inertia of the system with respect to the ζ-axis for the given q_i and the required form of the free surface of the liquid. Varying the integrand and using the equation of continuity and the boundary conditions for the liquid, we find that (3.18) is possible only if the free liquid surface has the form

$$\frac{k_0^2}{2S^2}(\xi^2+\eta^2)+U_2(\xi,\eta,\zeta)=\text{const.} \tag{3.19}$$

For steady-state motion of the system, when $q_i = 0$ $(i = 1, \ldots\ldots, n-1)$, equation (3.19) coincides with equation (3.10). Under fairly general assumptions on the smoothness of the cavity walls, the liquid level and the character of the mass forces acting on the liquid, it can be shown /24/ that if the free surface of the liquid is defined by (3.19) for given fixed q_i, then W has a minimum.

For any given values of q_i in the region (3.17), a solid body with a liquid-filled cavity can be associated with a certain solid body, which we call transformed, consisting of the given solid body and the frozen liquid with free surface (3.19). Then according to the above, W is a minimum for the transformed body in comparison with all possible free surfaces of the liquid sufficiently close to (3.19).

Theorem 3.2. *In order that W have a minimum in steady-state motion of a solid body with a liquid-filled cavity, it is necessary and sufficient that W have a minimum for $q_i = 0$ for the transformed solid bodies in the region (3.17).*

Thus, the minimum problem for W is reduced to the minimum problem for a function of a finite number of variables — to the case of W for a solid body with a liquid bounded by the cavity walls σ_1 and the free surface (3.19).

Let us find the variation of W for the transformed solid body in passing from a position corresponding to steady-state motion of the system for $q_i = 0$ to a perturbed position in the region (3.17). We can conceive of this transition as proceeding in two stages: 1) displacement of the whole system, as a single rigid body, to a perturbed position; 2) deformation of the form f_0 of the liquid (by covering its free surface with a layer τ_1 of zero volume) into a form f with free surface (3.19).

The increment of W may then be represented in the form

$$\Delta W=\Delta_1 W+\Delta_2 W, \tag{3.20}$$

where Δ_1 denotes the increment when the whole system is displaced to the perturbed position as a single rigid body, and Δ_2 is the increment due to the subsequent deformation of the free surface of the liquid into the surface (3.19). Similarly, $\Delta S = \Delta_1 S + \Delta_2 S$.

Up to second-order accuracy in q_i, we have

$$\Delta_1 W=\frac{1}{2}\sum_{i,j=1}^{[n-1]}\left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0 q_i q_j+\ldots,$$

$$\Delta_2 W=-\rho\int_{\tau}\left[\frac{1}{2}\omega_0^2(\xi^2+\eta^2)+U_2\right]d\tau+\frac{\omega_0^2}{2S_0}[(\Delta_2 S)^2+2\Delta_1 S\cdot\Delta_2 S]+\ldots,$$

where the zero index refers to the unperturbed position of the system.

To calculate $\Delta_2 W$, it is convenient to introduce a moving coordinate system xyz, rigidly attached to the solid body, whose z-axis in the unperturbed position coincides, say, with the ζ-axis.

The integrand in the expression for $\Delta_2 W$, in terms of the variables x, y, z, will be denoted by $\Phi(x, y, z, q_l)$. The equation of the free surface of the solidified liquid (3.10) in terms of the variables x, y and z is

$$\Phi(x, y, z, 0) = \frac{1}{2}\omega_0^2(x^2+y^2) + U_2(x, y, z) = c. \tag{3.21}$$

The value of the constant c is determined by the amount of liquid in the cavity. We assume that equation (3.21) has a unique solution with respect to one of the variables x, y, and z (for example, with respect to z); for this it is sufficient that at no point of the surface (3.21) $\partial\Phi/\partial z \neq 0$. Let us denote by Q the region of the xy plane bounded by the projection of a closed curve representing the locus of the intersection points of the surface (3.21) with the cavity walls σ_1.

Expressed in terms of the moving axes, the surface (3.19) assumes the form

$$\Phi_1(x, y, z, q_l) = c_1, \tag{3.22}$$

where the constant $c_1 = c + \Delta c$ is determined from the equality of the liquid volumes in the cavity, having free surfaces (3.21) and (3.22). The latter condition is equivalent to the fact that the volume of the deforming layer τ_1 is zero:

$$\int\limits_{\tau_1} d\tau = 0.$$

In the first approximation this equation is equivalent to the equation

$$\iint\limits_{Q} dx\,dy \int\limits_{z_0}^{z_1} dz = 0,$$

where z_0 and z_1 denote, respectively, the values of the variable z for points of the surface (3.21) and (3.22). Replacing the variable z by a new variable $\mu = \Phi(x, y, z, q_l) - c$, the above equation can be represented with the same accuracy in the form

$$\iint\limits_{Q} \left(\frac{\partial z}{\partial \Phi}\right)_0 (\mu_1 - \mu_0)\,dx\,dy = 0, \tag{3.23}$$

where, to within first-order accuracy,

$$\begin{aligned} \mu_0 &= \Phi(x, y, z_0, q_l) - c = \sum_{l=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_l}\right)_0 q_l + \ldots, \\ \mu_1 &= \Phi(x, y, z_1, q_l) - c = \Delta c + \frac{\omega_0^2}{S_0}(x^2+y^2)\Delta S + \ldots, \end{aligned} \tag{3.24}$$

since in the first approximation the functions $\Phi(x, y, z, q_i)$ and $\Phi_1(x, y, z, q_i)$ differ only by the term $\frac{\omega_0^2}{S_0}(x^2+y^2)\Delta S$. Substituting the values of μ_0 and μ_1 in (3.23) we obtain a linear equation connecting Δc with $\Delta_2 S$. A second similar equation is obtained by calculating, in the first approximation, the quantities

$$\Delta_2 S = \rho\int_{\tau_1}(\xi^2+\eta^2)\,d\tau = \rho\iint_Q\left(\frac{\partial z}{\partial\Phi}\right)_0(x^2+y^2)(\mu_1-\mu_0)\,dx\,dy. \qquad (3.25)$$

Equations (3.23) and (3.25), in conjunction with (3.24), yield Δc and $\Delta_2 S$ as functions of q_i. We thus obtain

$$\Delta_2 W = -\frac{1}{2}\rho\iint_Q\left(\frac{\partial z}{\partial\Phi}\right)_0(\mu_1^2-\mu_0^2)\,dx\,dy + \frac{\omega^2}{2S_0}[(\Delta_2 S)^2 + 2\Delta_1 S\cdot\Delta_2 S] + \ldots$$

By formula (3.20), ΔW is represented as a quadratic form in the variables $q_1, \ldots, q_{n-1}$. The conditions for positiveness of the latter are conditions for a minimum of W for a solid body with a liquid-filled cavity in a field of external forces with potential energy V.

If ΔW is not definite, W does not have a minimum.

Setting $k_0 = 0$ in these formulas, we obtain /23/ the solution of the problem of minimum potential energy V for a solid body with a liquid. We mention that theorems analogous to 3.1 and 3.2, referring only to the equilibrium of a system subject to a gravitational force, were first established in the linear formulation by N.N. Moiseev /25/.

Example 3 /24/. In a uniform gravitational field, consider a solid body with a fixed point, having a cavity partially filled with liquid. Let us investigate the stability of uniform rotation of the whole system as a single rigid body about the principal central axis of inertia z, which coincides with the vertical ζ.

We assume the region Q to be an annulus bounded by circles with radii R_1 and R_2 $(R_1 > R_2)$. The conditions for minimum W reduce in this case to the single inequality

$$(C_0 - A_0)\omega^2 - Mgz_0 - a > 0 \quad (A_0 > B_0),$$

where

$$a = \pi\rho g\int_{R_2}^{R_1}\left[\frac{\omega^2}{g^2}\left(\frac{\omega^2}{2}r^2 - c\right) + 1\right]^2 r^2\,dr.$$

For other examples, see /23, 24, 26/.

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STABILITY OF SYSTEMS WITH DELAY

This paper is a review of the literature on the stability of motion of systems described by differential equations with aftereffect or time delay. Our discussion is centered around generalizations of the well-known classical theory of Lyapunov stability for systems described by ordinary differential equations /1, 2, 3, 4/. To this category belong investigations by means of functionals, the general approach to the analysis of delay equations in function spaces, stability in the first approximation, the theory of the critical cases, and stability theory of linear and nonlinear periodic systems. We refer in the main to published work of Soviet and non-Soviet authors and to some new results of a general character in stability theory of periodic systems.

Interest in the development of the theory of systems with delay, particularly stability theory of such systems, stems not only from the desire to generalize Lyapunov's method to increasingly more general objects of investigation, but also from practical needs. It is well-known that automatic control systems contain units with time delay. The delay may be in the controlled object, in the controller and in the feedback. Delay exists in servomechanisms with long communication lines. Finally, delay differential equations are convenient tools for mechanical and electrical systems with distributed elements or elements with many degrees of freedom. Recently, delay differential equations have been applied in control theory, when an uncontrolled system of ordinary differential equations becomes controllable if it is capable of storing information over some time interval; the motion of the system is then described by a system with aftereffect.

The development of the theory of delay systems and, in particular, of the theory of stability, is thus conditioned by practical needs.

1. Methods of investigation. General approach to the problem

Consider a general system of delay differential equations in the form

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\vartheta)\, d\eta_{ij}(\vartheta) + X_i(x_1(t+\vartheta), \ldots, x_n(t+\vartheta)) \quad (i=1, \ldots n),$$

where the integrals are Stieltjes integrals /4, p. 159/, $\eta_{ij}(\vartheta)$ are functions of bounded variation, $X_i(x_1(\vartheta), \ldots, x_n(\vartheta))$ are nonlinear functionals defined on piecewise-continuous functions $\{x_i(\vartheta)\}$ of ϑ in the interval $\tau \leqslant \vartheta \leqslant 0$, and representing nonlinear perturbations.

More precisely, the X_i satisfy Lipschitz conditions in x_i:

$$|X_i(x_1''(\vartheta),\dots,x_n''(\vartheta)) - X_i(x_1'(\vartheta),\dots,x_n'(\vartheta))| \leqslant L\|x''(\vartheta)-x'(\vartheta)\|_\tau,$$

$$\|x(\vartheta)\|_\tau = \sup\{|x_1(\vartheta)|,\dots,|x_n(\vartheta)|,\ -\tau\leqslant\vartheta\leqslant 0\}, \qquad (1.2)$$

$$L = L_1\{\|x''(\vartheta)\|_\tau + \|x'(\vartheta)\|_\tau\}^{\alpha_1}.$$

where L_1, α_1 are positive numbers, $X_i(0,\dots,0) = 0$.

The motion $x = 0$ will be called the unperturbed motion of system (1.1). Definitions of stability, asymptotic stability and instability of unperturbed motion are given as usual in terms of the norm $\|x(\vartheta)\|_\tau$.

It is well-known that Lyapunov's second method (the method of Lyapunov functions) is the basic method for solving stability problems. The first attempts to apply it to delay systems were unsuccessful /5/.

A modification of the method of Lyapunov functions, in which the function is required to be negative definite on a more restricted set of curves satisfying some additional condition, turned out to be more effective /6, 7/. The main shortcoming of Lyapunov functions in investigations of the stability of delay systems is their nonuniversality, and the irreversibility of the stability and asymptotic stability theorems.

The specific features of equations with time delay or of more general delay differential equations were first taken into account by replacing Lyapunov functions by Lyapunov functionals defined in some function space, such as C (the space of continuous functions). Applied to solutions in the space of continuous functions, the theorems of Lyapunov's second method turned out to be a universal means of investigation, due to their reversibility /4/.

A number of fundamental results concerning the stability of systems with aftereffect were obtained by the universal method of Lyapunov "functionals" /4/. Delay differential equations (i.e., equations with aftereffect) are functional equations, since they determine the time derivatives of the required quantities $x_s(t)$ at an instant t as functions of their values not only at the instant t, but also at instants $t-\tau_j$ preceding t in the delay interval $[t-\tau, t]$ (the time derivatives of the quantities $x_s(t)$ are functionals on the interval $\{x_s(t+\vartheta),\ -\tau\leqslant\vartheta\leqslant 0\}$, τ being the delay). Therefore, as an element of the solution it is natural to take a segment of the integral trajectory of the delay system, and then to consider the solution itself in the space of continuous functions $C[-\tau,0]$, $x(\vartheta) = \{x_i(\vartheta),\ i = 1,\dots,n,\ -\tau\leqslant\vartheta\leqslant 0\}$ with norm $\|x(\vartheta)\|_\tau = \sup(|x_1(\vartheta)|,\dots,|x_n(\vartheta)|,\ -\tau\leqslant\vartheta\leqslant 0)$.

This approach to delay systems was suggested in /4, 8/.

Thus, as an element of the solution of system (1.1) we shall take a vector-segment of the trajectory $x(x_0(\omega), t_0, t+\vartheta)$, $-\tau\leqslant\vartheta\leqslant 0$, rather than the vector-function of the time $x(x_0(\vartheta), t_0, t)$. Calculating the first derivative of $x(x_0(\vartheta), t_0, t+\vartheta)$ with respect to the time t for $\vartheta < 0$, we find

$$\frac{dx_s(x_0(\vartheta), t_0, t+\vartheta)}{dt} = \frac{dx_s(x_0(\vartheta), t_0, t+\vartheta)}{d\vartheta};$$

$$\Delta t\to+0, \quad -\tau\leqslant\vartheta<0.$$

The system of equations (1.1) in the functional space is therefore associated with an equivalent system of "ordinary differential equations" with an operator right-hand side /4, p. 162; 8/:

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + R(x_t(\vartheta)), \tag{1.2}$$

where

$$x_t(\vartheta) = x(t+\vartheta) = \{x_i(t+\vartheta), \quad -\tau \leqslant \vartheta \leqslant 0\},$$

$$Ax(\vartheta) = \begin{cases} \dfrac{dx_k(\vartheta)}{d\vartheta}, \quad -\tau \leqslant \vartheta < 0 \\ \qquad\qquad\qquad\qquad (k = 1, \ldots, n) \\ \sum\limits_{l=1}^{n} \int\limits_{-\tau}^{0} x_l(\vartheta)\, d\eta_{kl}(\vartheta), \quad \vartheta = 0 \end{cases} \tag{1.3}$$

$$R(x(\vartheta)) = \begin{cases} 0, \quad -\tau \leqslant \vartheta < 0 \\ X_k(x_1(\vartheta), \ldots, x_n(\vartheta)), \quad \vartheta = 0. \end{cases} \tag{1.4}$$

The fact that A is an unbounded linear operator is no obstacle to the use of system (1.2) as a convenient working tool. This approach to delay differential equations and the method of functionals has been found effective in investigating delay systems, and has led to considerable advances in stability theory for this type of system /4, 8/.

As an example of this approach, we discuss in more detail some supplements to the theory of linear stationary delay equations.

2. The theory of linear delay differential equations

Consider a system of linear delay differential equations in the form

$$\frac{dx_s(t)}{dt} = \sum_{l=1}^{n} \int\limits_{-\tau}^{0} x_l(t+\vartheta)\, d\eta_{sl}(\vartheta) \quad (s = 1, \ldots, n), \tag{2.1}$$

where the integrals on the right-hand side are again Stieltjes integrals, $\eta_{sl}(\vartheta)$ are functions of bounded variation. A particular case of system (2.1) is the delay system

$$\frac{dx_s(t)}{dt} = \sum_{l=1}^{n} (a_{sl} x_l(t) + b_{sl} x_l(t-\tau)) \quad (s = 1, \ldots, n). \tag{2.2}$$

The latter is (2.1) with $\eta_{sl}(0) = a_{sl}$, $\eta_{sl}(-\tau) = b_{sl}$, $\eta_{sl}(\vartheta) = 0$, $-\tau < \vartheta < 0$. Linear delay equations are studied in detail in /9/.

The system of "ordinary differential equations" with operator right-hand side equivalent to (2.1) is:

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta), \tag{2.3}$$

where the operator A is defined by (1.3). For system (2.2) the operator A has the form $Ax(\vartheta) = \{dx_k(\vartheta)/d\vartheta$ for $\tau \leqslant \vartheta < 0$, $ax(0) + bx(-\tau)$ for $\vartheta = 0\}$.

Together with system (2.3), we consider the adjoint system /10, 11, 12/

$$\frac{dy_t(\vartheta)}{dt} = -A^* y_t(\vartheta), \tag{2.4}$$

where

$$y_t(\vartheta) = y(t+\vartheta) = \{y_i(t+\vartheta),\ \tau > \vartheta > 0\};$$

$$-A^* y(\vartheta) = \begin{cases} \dfrac{dy_k(\vartheta)}{d\vartheta}, & \tau > \vartheta > 0 \\ & \qquad (k = 1, \ldots, n) \\ -\sum\limits_{j=1}^{n} \int\limits_{\tau}^{0} y_j(\vartheta)\, d\eta_{jk}(-\vartheta), & \vartheta = 0 \end{cases} \tag{2.5}$$

For system (2.2), the adjoint operator $-A^*$ is defined by

$$-A^* y(\vartheta) = \left\{\frac{dy_k(\vartheta)}{d\vartheta},\quad \tau > \vartheta > 0,\quad -a' y(0) - b' y(\tau)\ \text{ for } \vartheta = 0\right\}.$$

System (2.4) corresponds to a system of differential equations with time advance of the form

$$\frac{dy_s(t)}{dt} = -\sum_{j=1}^{n} \int\limits_{\tau}^{0} y_j(t+\vartheta)\, d\eta_{js}(-\vartheta) \quad (s = 1, \ldots, n). \tag{2.6}$$

The spectrum of the operator A consists of eigenvalues λ_j $(j = 1, 2, 3, \ldots)$ — the roots of the characteristic equation

$$\Delta(\lambda) \equiv \left| -E\lambda + \int\limits_{-\tau}^{0} e^{\lambda\vartheta}\, d\eta(\vartheta) \right| = 0. \tag{2.7}$$

The eigenvalues of the operator $-A^*$ satisfy the relation $\mu_\sigma = -\lambda_\sigma$, $(\sigma = 1, 2, \ldots)$.

We define the scalar product of two vectors

$$x(\vartheta)\ (-\tau < \vartheta < 0) \quad \text{and} \quad y(\vartheta)\ (\tau > \vartheta > 0):$$

$$(x(\vartheta), y(\vartheta)) = \sum_{j=1}^{n} x_j(0)\, y_j(0) - \sum_{i,j=1}^{n} \int\limits_{-\tau}^{0} \left[\int\limits_{0}^{\vartheta} x_i(\vartheta)\, y_j(-\vartheta+\xi)\, d\xi\right] d\eta_{ji}(\vartheta). \tag{2.8}$$

A direct calculation establishes the identity

$$(Ax(\vartheta), y(\vartheta)) = (x(\vartheta), A^* y(\vartheta)) \tag{2.9}$$

for any elements $x(\vartheta)$ and $y(\vartheta)$.

It follows that for any solution of system (2.4), proceeding in the direction of increasing time t, the expression

$$(x(\vartheta),\ y(t+\vartheta)) = C \tag{2.10}$$

is a first integral of system (2.3). Here we have an analogy with the theory of ordinary linear differential equations with constant coefficients.

There exists an integer n_0 such that the multiplicity of any root of the characteristic equation (2.7) does not exceed n_0. Let λ_j be a root of the characteristic equation (2.7). Then the equations

$$(A-\lambda_j I)^{n_0}x(\vartheta)=0, \quad (-A^{*}-\lambda_j I)^{n_0}y(\vartheta)=0 \tag{2.11}$$

(where I is the identity operator, $Ix(\vartheta) = x(\vartheta)$) have an equal number of linearly independent nontrivial solutions—the root elements $x_j(\vartheta)$ of the operator A and the root elements $y_j(\vartheta)$ of the operator $-A^{*}$. The solutions of equations (2.11) for $n_0 = 1$ are called eigenelements, all the other root elements are called associated elements. Let the set of root elements of the operator A for the roots $\lambda_1, \ldots, \lambda_N$ be $x_1(\vartheta), \ldots, x_N(\vartheta)$, and the set of root elements of the operator $-A^{*}$ corresponding to the same roots be $y_1(\vartheta), \ldots, y_N(\vartheta)$.. It can be shown that these elements can always be chosen to satisfy the following conditions.

If $x_j(\vartheta)$ is an eigenelement of the operator A with no associated elements, then

$$(x_j(\vartheta),\ y_\sigma(\vartheta))=\begin{cases}1, & j=\sigma\\ 0, & j\neq\sigma.\end{cases} \tag{2.12}$$

If the elements $x_j(\vartheta), x_{j+1}(\vartheta), \ldots, x_{j+m}(\vartheta)$ form a Jordan chain ($x_j(\vartheta)$ is an eigenelement, and $x_{j+1}(\vartheta), \ldots, x_{j+m}(\vartheta)$ associated elements), then a Jordan chain $y_j(\vartheta), \ldots, y_{j+m}(\vartheta)$ corresponds to the adjoint operator and we have

$$\begin{gathered}(x_{j+k}(\vartheta),\ y_{j+m-\sigma}(\vartheta))=\begin{cases}1 & \sigma=k,\quad 0\leqslant\sigma\leqslant m\\ 0 & \sigma\neq k,\quad 0\leqslant\sigma\leqslant m,\end{cases}\\ (x_{j+k}(\vartheta),\ y_\sigma(\vartheta))=0 \quad (\sigma<j,\ \sigma>j+m);\end{gathered} \tag{2.13}$$

and

$$\begin{gathered}(A-\lambda_j I)\,x_j(\vartheta)=0\\ (A-\lambda_j I)\,x_{j+k}(\vartheta)=x_{j+k-1}(\vartheta) \quad (k=1,\ldots,m).\end{gathered} \tag{2.14}$$

Suppose that $\lambda_1, \ldots, \lambda_N$ are all the roots of the characteristic equation (2.7) satisfying $\operatorname{Re}\lambda_j > -\alpha$, $(j = 1, \ldots, N)$, where α is some positive number (counting each λ_j with its multiplicity). Let us define $N = N(\alpha)$ functionals

$$f_j(x(\vartheta))=(x(\vartheta),\ y_j(\vartheta)) \quad (j=1,\ldots,N). \tag{2.15}$$

The conditions

$$f_j(x(\vartheta))=0, \quad (j=1,\ldots,N) \tag{2.16}$$

determine a subspace $\text{Л}(\alpha)$ in the space $C\,[-\tau,0]$. An arbitrary element $x(\vartheta)$ of the space $C\,[-\tau,0]$ can be represented in the form

$$x(\vartheta)=\sum_{\sigma=1}^{N} y_\sigma\cdot x_\sigma(\vartheta)+z(\vartheta). \tag{2.17}$$

where

$$y_\sigma = f_\sigma(x(\vartheta)), \tag{2.18}$$

in the case (2.12), and

$$y_{\sigma+k} = f_{\sigma+m-k}(x(\vartheta)) \quad (k = 1, \ldots, m), \tag{2.19}$$

in the case (2.13).

Now the system of equations (2.3) may be expressed in terms of the new variables $y_1, \ldots, y_N$ and $Z(\vartheta)$, assuming the form

$$\frac{dy_j}{dt} = \lambda_j y_j, \quad \text{in the case (2.12)}$$

$$\left.\begin{aligned} &\frac{dy_{j+m}}{dt} = \lambda_j y_{j+m}, \\ &\frac{dy_{j+m-1}}{dt} = \lambda_j y_{j+m-1} - y_{j+m}, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\frac{dy_j}{dt} = \lambda_j y_j - y_{j+1}, \end{aligned}\right\} \quad \text{in the case (2.13)} \tag{2.20}$$

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta), \quad z_t(\vartheta) \in \text{Л}(\alpha). \tag{2.21}$$

The spectrum of the operator A restricted to $\text{Л}(\alpha)$ consists of the eigenvalues λ_σ, $\sigma = N + 1. \ldots,$ with the exception of the first $N : \lambda_1, \ldots, \lambda_N$.

3. The theory of linear periodic delay systems

Consider systems whose motion is described by delay differential equations with periodic coefficients, of the form

$$\frac{dx_s}{dt} = F_s(t, x_1(t+\vartheta), \ldots, x_n(t+\vartheta)) \quad (s = 1, \ldots, n), \tag{3.1}$$

where

$$F_s(t, x_1(\vartheta), \ldots, x_n(\vartheta)) = \sum_{j=1}^{n} \Big\{ p_{sj}(t) x_j(0) +$$
$$+ \sum_{\sigma=1}^{K} q_{\sigma sj}(t) x_j(-\tau_\sigma) + \int_{-\tau}^{0} f_{sj}(t, \xi) d\xi \Big\}, \quad (s = 1, \ldots, n).$$

Here p_{sj}, $q_{\sigma sj}$ are periodic continuous functions of the time t with period ω, the functions $f_{sj}(t, \xi)$ are continuous in t and ξ for $\tau \leqslant \xi \leqslant 0$, $-\infty < t < +\infty$ and periodic in t with period ω, $\tau > \tau_\sigma$ is the delay.

This type of system was considered in /13, 14, 15, 16, 17/. In the space of continuous functions $C[-\tau, 0]$ with the same norm $\|x(\vartheta)\|_\tau$, equations (3.1) are equivalent to the following system of ordinary differential equations with operator right-hand side:

$$\frac{dx_t(\vartheta)}{dt} = P(t)\,x_t(\vartheta), \tag{3.2}$$

where $x_t(\vartheta) = x(t+\vartheta) = \{x_s(t+\vartheta),\ -\tau \leqslant \vartheta \leqslant 0,\ s = 1, \ldots, n\}$, and the operator $P(t)$ is defined by

$$P(t)\,x(\vartheta) = \begin{cases} \dfrac{dx_s(\vartheta)}{d\vartheta}, & -\tau \leqslant \vartheta < 0 \\ F_s(t, x_1(\vartheta), \ldots, x_n(\vartheta)), & \vartheta = 0. \end{cases} \quad (s = 1, \ldots, n) \tag{3.3}$$

Let $x(x_0(\vartheta), t_0, t)$ be a solution of system (3.1) with initial function $x_0(\vartheta)$ at the instant t_0, $x(x_0(\vartheta), t_0, t_0 + \vartheta) = x_0(\vartheta)$. Then $x_t(\vartheta) = x(x_0(\vartheta), t_0, t + \vartheta)$, $-\tau \leqslant \vartheta \leqslant 0$ defines a solution of system (3.2) with the initial function $x_0(\vartheta)$. For fixed t, the element $x_t(\vartheta)$ of this solution can be regarded as the image of the initial element $x_0(\vartheta)$ under the mapping

$$x_t(\vartheta) = T(t, t_0)\,x_0(\vartheta), \tag{3.4}$$

where $T(t, t_0)$ is a linear operator, $T(t_0, t_0) = I$.

It may be shown that the eigenvalues of a completely continuous operator $T(t_0 + \omega, t_0)$ are independent of t_0. The spectral radius r_t of the operator $T(t_0 + \omega, t_0)$ determines the stability or instability of the motion $x = 0$. If r_t is greater than unity, the motion $x = 0$ of the system (3.1) is unstable. If r_t is smaller than unity, the motion $x = 0$ is asymptotically stable. The analytical form of the eigenelements and the associated elements of the operator is obtained. The corresponding particular solutions of (3.1) are continuable to the whole time axis and have the same analytical form. Thus, the eigenvector $x_e(\vartheta)$ corresponding to the eigenvalue ρ_e has the form

$$x_e(\vartheta) = \rho_e^{\vartheta/\omega}\Phi_e(\vartheta), \quad -\tau \leqslant \vartheta \leqslant 0, \tag{3.5}$$

where $\Phi_e(\vartheta)$ is a periodic vector with period ω, for ϑ in the interval $(-\infty, +\infty)$. The $\Phi_e(\vartheta)$ have derivatives of arbitrary order with respect to ϑ.

Consider the adjoint system of differential equations with time advance /16, 17/

$$\frac{dy_s(t)}{dt} = -F_s^*(t, y_1(t+\vartheta), \ldots, y_n(t+\vartheta)), \quad (s = 1, \ldots, n), \tag{3.6}$$

where

$$-F_s^*(t, y_1(\vartheta), \ldots, y_n(\vartheta)) = -\sum_{j=1}^{n} p_{js}(t)\,y_j(0) -$$
$$-\sum_{j=1}^{n}\sum_{\sigma=1}^{k} q_{\sigma js}(t+\tau_\sigma)\,y(\tau_\sigma) - \sum_{j=1}^{n}\int_{-\tau}^{0} f_{js}(t-\xi, \xi)\,y_j(-\xi)\,d\xi \quad (s = 1, \ldots, n).$$

Here p, q, f are the same as in (3.1). The equivalent system of linear ordinary differential equations with operator right-hand side in the functional space $C[-\tau, 0]$ is /16/:

$$\frac{dy_t(\vartheta)}{dt} = -P^*(t)\,y_t(\vartheta), \tag{3.7}$$

where

$$y_t(\vartheta) = y(t+\vartheta) = \{y_s(t+\vartheta),\quad \tau > \vartheta > 0,\quad s = 1, \ldots, n\},$$
$$-P^*y(\vartheta) = \left\{\frac{dy_s(\vartheta)}{d\vartheta}\ \text{for}\ \tau > \vartheta > 0,\ F^*(t, y(\vartheta))\ \text{for}\ \vartheta = 0\right\}.$$

Let us denote by $y(y_0(\vartheta), t_0, t)$ a solution of system (3.6) with initial function $y_0(\vartheta), \tau > \vartheta > 0$ at the instant t_0 for $t \leqslant t_0$; then $y_t(\vartheta) = y(y_0(\vartheta), t_0, t+\vartheta), \tau > \vartheta > 0$ is a solution of system (3.7). For fixed t, the element of this solution can be regarded as the image of the element $y_t(\vartheta)$ under the mapping $y_0(\vartheta)$

$$y_t(\vartheta) = T^*(t, t_0)y_0(\vartheta),\quad t < t_0. \tag{3.8}$$

Again the eigenvalues of the operator $T^*(t_0 - \omega, t_0)$ do not depend on t_0 and coincide with those of the operator $T(t_0 + \omega, t_0)$. The structure of the eigenvectors and the associated vectors of the operator $T^*(t_0 - \omega, t_0)$ is identical with that of the analogous vectors for the operator $T(t_0 + \omega, t_0)$. The corresponding particular solutions are continuable not only to $t < t_0$, but also to $t > t_0$.

We introduce the notation

$$\begin{aligned}(x(\vartheta), y(\vartheta), t) = {} & \sum_{j=1}^{n} x_j(0)y_j(0) + \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\sigma=1}^{k}\int_0^{\tau_\sigma} x_l(\xi-\tau)\cdot y_j(\xi)\,q_{\sigma jl}(t+\xi)\,d\xi - \\ & - \sum_{j=1}^{n}\sum_{l=1}^{n}\int_{-\tau}^{0}\left[\int_{-\vartheta}^{0} x_l(\xi+\vartheta)\,y_j(\xi)\,f_{jl}(t+\xi, \vartheta)\,d\xi\right]d\vartheta.\end{aligned} \tag{3.9}$$

For any particular solution $y_t(\vartheta)$ of system (3.9), continuable to $t > t_0$, the expression

$$(x_t(\vartheta), y_t(\vartheta), t) = \mathrm{const} \tag{3.10}$$

is a first integral of the system (3.2).

Suppose that all the eigenvalues ρ_j satisfying the condition $|\rho_j| > \varepsilon$ (ε arbitrary) are simple and let their number be $N(\varepsilon)$. The more general case is considered analogously /16/. Consider eigenvectors $x_j(\vartheta), y_j(\vartheta)$, ($j = 1, \ldots, N(\varepsilon)$) of the operators $T(t_0 + \omega, t_0)$, $T^*(t_0 - \omega, t_0)$. Let $x_j(\vartheta)$ and $y_j(\vartheta)$ correspond to the eigenvalue ρ_j; then

$$(x_j(\vartheta), y_j(\vartheta), t_0) = \begin{cases}1, & j = k\\ 0, & j \neq k.\end{cases} \tag{3.11}$$

Let $x_j^{(t)}(\vartheta)$ and $y_j^{(t)}(\vartheta)$ be particular solutions of systems (3.1) and (3.9) with initial functions $x_j(\vartheta)$ and $y_j(\vartheta)$, respectively (for $t = t_0$). Every element $x(\vartheta)$ of the space $C[-\tau, 0]$ can be represented uniquely as a sum

$$x(\vartheta) = \sum_{j=1}^{N} a_j x_j(t - t_0 + \vartheta)\,\rho_j^{-\frac{t-t_0}{\omega}} + z(\vartheta), \tag{3.12}$$

where

$$a_j = \left(x(\vartheta),\ y_j(t-t_0+\vartheta)\,\rho_j^{\frac{t-t_0}{\omega}},\ t\right) \quad (j=1,\ldots,N), \tag{3.13}$$

$$\left(z(\vartheta),\ y_j(t-t_0+\vartheta)\,\rho_j^{\frac{t-t_0}{\omega}},\ t\right)=0, \quad (j=1,\ldots,N). \tag{3.14}$$

Thus, one can always pass from the variable $x(\vartheta)$ in the system (3.2) to the scalar variables $a_1,\ldots,a_N$ and the variable $z(\vartheta)$, which belongs to the subspace (3.14). In terms of these variables (3.2) corresponds to the system

$$\frac{da_j}{dt}=\frac{1}{\omega}\log\rho_j\cdot a_j \quad (j=1,\ldots,N); \tag{3.15}$$

$$\frac{dZ_t(\vartheta)}{dt}=P(t)\,z_t(\vartheta), \quad z_t(\vartheta)\in(3.16). \tag{3.16}$$

Formula (3.12) may then be interpreted as follows. In the space $C[-\tau,0]$ one always can find a periodically shifting N-dimensional basis $\{x_j(t-t_0+\vartheta)\times\bar{\rho}_j^{\frac{t-t_0}{\omega}}\}$ $(j=1,\ldots,N)$, (with period ω). The coordinates a_j describing the motion of the system in the corresponding N-dimensional subspace of $C[-\tau,0]$ then satisfy a system of ordinary differential equations with constant coefficients (3.15). The component $z_t(\vartheta)$ in (3.12) belongs to the subspace (3.14) and decreases exponentially without limit, with exponent $\beta\leqslant\frac{1}{\omega}\log\varepsilon\ (\varepsilon<1)$.

4. Stability in the first approximation and stability investigations in the critical cases

Consider again system (1.1), together with the so-called first approximation system (2.1) resulting from system (1.1) by putting $X_2\equiv 0$. It was shown in /4, 18/ that if all the roots of the characteristic equation (2.7) have negative real parts the unperturbed motion $x=0$ of systems (2.1) and (1.1) is asymptotically stable, irrespective of the form of the nonlinear parts X_2 in (1.1). If at least one of the roots of equation (2.7) has a positive real part, the unperturbed motion $x=0$ of systems (2.1) and (1.1) is unstable, irrespective of the form of the nonlinear parts X_2/10/. Thus the stationary delay system (1.1) is completely analogous to ordinary differential equations as regards the problem of stability in the first approximation. The same is true of nonlinear periodic systems of type (1.1) with first approximation systems of type (3.1). The first proposition, concerning asymptotic stability, is proved in /4, 19/. The second (instability) is proved, as in /10/, on the basis of the decomposition indicated in /16/.

When the characteristic equation may have zero or purely imaginary roots, we have the so-called critical case. Critical cases of one zero and a pair of purely imaginary roots are considered, for stationary systems of type (1.1), in /20/.

Let us investigate stability in the critical case of a single zero root $\lambda_1=0$ in more detail. It is assumed that all the other roots of equation (2.7) have negative real parts. The technique is a generalization of Lyapunov's method, which he developed for systems described by ordinary differential equations. In the case under consideration $\Delta(0) = 0$, but $\Delta'(0) \neq 0$. Let $\Delta_{kj}(0)$ denote the cofactor of the element at the intersection of the k-th row and the j-th column of the determined $\Delta(0)$ of (2.7). Let $\Delta_{k_1 l_1}(0) \neq 0$, $y_1(\vartheta) = \{\Delta_{jk_1}(0),\ j = 1, \ldots, n;\ \tau > \vartheta > 0\} = \mathrm{const}$. Consider the functional $f(x(\vartheta)) = (x(\vartheta), y_1(\vartheta))$, and the vector $x_1(\vartheta)$ in $C[-\tau, 0]$:

$$x_1(\vartheta) = \{\Delta_{l_1, j}(0)\}\,(\Delta_{l_1 k_1}(0)\,\Delta'(0))^{-1} = \mathrm{const}(\vartheta), \quad -\tau < \vartheta < 0. \tag{4.1}$$

Any element $x(\vartheta) \in C[-\tau, 0]$ can be represented uniquely in the form

$$x(\vartheta) = x_1(\vartheta)\,y + z(\vartheta). \tag{4.2}$$

Equation (2.3) becomes

$$\begin{gathered} dy/dt = 0, \\ dz_t(\vartheta)/dt = Az_t(\vartheta), \quad f[z_t(\vartheta)] = 0, \end{gathered} \tag{4.3}$$

while system (1.2) becomes:

$$\begin{gathered} dy/dt = Y(y, z(\vartheta)), \\ dz_t(\vartheta)/dt = Az_t(\vartheta) + Z(y, z_t(\vartheta), \vartheta), \quad f[z_t(\vartheta)] = 0. \end{gathered} \tag{4.4}$$

Here $Y(y, z(\vartheta))$ is a functional defined by $Y(y, z(\vartheta)) = f[R]$, and the operator

$$Z(y, z(\vartheta), \vartheta) = R(x_1(\vartheta)\,y + z(\vartheta)) - x_1(\vartheta)\,Y(y, z(\vartheta)). \tag{4.5}$$

Consider the function

$$Y(y, 0) = gy^m + \ldots, Z(y, 0, \vartheta) = g_1(\vartheta)\,y^{m_1} + \ldots \tag{4.6}$$

Let $m_1 > m$. Then if m is odd and $g < 0$, the motion $x = 0$ is asymptotically stable. If m is even or odd, but $g > 0$, the motion $x = 0$ is unstable.

The condition $m_1 > m$ can be ensured by the substitution $z(\vartheta) = z_1 + u(\vartheta, y)$, where $u(\vartheta, y)$ is a solution of the equation

$$Au(\vartheta, y) + Z(y, u(\vartheta, y), \vartheta) = 0, \tag{4.7}$$

which satisfies the condition $f[u(\vartheta, y)] = 0$. On the plane $f[x(\vartheta)] = 0$ equation (4.7) has a unique solution $u(\vartheta, y)$, which is analytic with respect to y and differentiable with respect to ϑ.

This type of transformation always ensures the condition $m_1 > m$, unless we have the singular case $Y(y, u(\vartheta, y), \vartheta) \equiv 0$. In this case the unperturbed motion $x = 0$ is stable /20/.

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CELESTIAL BALLISTICS

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OPTIMIZATION PROBLEMS IN THE MECHANICS OF LOW-THRUST SPACE FLIGHT

The development of space rocket technology is creating new branches of mechanics. The idea of using rocket engines for space flight, which took shape at the turn of the century, has stimulated the development of the mechanics of space flight (Meshcherskii, Tsiolkovskii, Tsander, Kosmodem'yanskii, Goddard, Hohmann, and others). This science studies the motion of space vehicles as bodies of variable mass in order to determine conditions for the delivery of a maximum payload. Here the type of propulsion system plays a decisive role.

The mechanics of high-thrust space flight studies the flight of rockets with heat engines, the name indicating their characteristic feature—small weight/thrust ratio; the thrust should be of the same order as the weight of the rocket.

At the turn of the century the principles of low-thrust flight (solar sail, electrical jet engines) were formulated. The outlook for these propulsion systems is determined by the small mass flow per unit thrust (for electrical jet engines) or the absence of mass flow (for solar sails). Recent technological progress has raised the problem of scientific and technological realization of these principles, and a new branch of mechanics has developed—the mechanics of low-thrust space flight.

The origin of the term "low-thrust" is connected with the high (at the present stage) specific gravity of the propulsion unit; vehicles with such engines need a thrust which is small in comparison with their weight and which operates over a prolonged time. Investigation has shown that, as far as the payload is concerned, vehicles with low-thrust engines generally have advantages over others in identical space maneuvers.

The present paper deals mainly with the optimization problem in the mechanics of low-thrust space flight. This consists in selecting optimal relations between the weight components of the vehicle and optimal control of the propulsion system, and determining the set of optimal flight trajectories. A more detailed exposition and an extensive bibliography can be found in the authors' survey /1/.

1. General formulation of the optimization problem in flight mechanics

In formulating the purpose of a space maneuver, the problem of its best realization arises. Usually it is required to deliver the maximum payload for given launching weight and fixed performance time. In the sequel we describe a general formulation of this problem, without referring it to specific propulsion systems.

We first enumerate the basic weight components of the vehicle: G_{m0}, $G_m(t)$ are the initial and current weights of the propellant; $G_{\varkappa 0}$, $G_{\varkappa}(t)$ are the initial and current weights of the propulsion system (if sections of the engine are jettisoned during the motion); G_p is the weight of the payload; G_0, $G(t)$ are the initial and current values of the total weight of the vehicle

$$G = G_m + G_{\varkappa} + G_p \qquad (1.1)$$

All the weight components not involved in the optimization conditions are usually included in the payload.

Let us establish the relation between the weight characteristics and the parameters of motion of the system, and determine the possibility of thrust control. An obvious relation between the mass flow q and the current weight $G_m(t)$ of the propellant is

$$\dot{G}_m = -gq. \qquad (1.2)$$

The weight $G_{\varkappa}$ of the propulsion system is a function of the extreme values $u_{j\,\max}$ and $u_{j\,\min}$ of the internal control parameters u_j of the engine

$$G_{\varkappa} = G_{\varkappa}(u_{j\max}, u_{j\min}), \qquad (1.3)$$

which control the thrust P and mass flow q:

$$P = P(u_j), \quad q = q(u_j); \quad (u_{j\min} \leqslant u_j \leqslant u_{j\max}). \qquad (1.4)$$

Relations (1.3) and (1.4) must be specified for each type of propulsion system.*

The motion in a gravitational field of the center of mass of a space-vehicle due to the thrust can be described by a vector differential equation

$$\mathbf{r} = (P/G)g\mathbf{l} + \mathbf{R}, \qquad (1.5)$$

where $\mathbf{r}(t)$ is the radius-vector in an inertial coordinate system, $\mathbf{R}(\mathbf{r}, t)$ is the gravitational acceleration vector, $\mathbf{l}$ is a unit vector in the thrust direction.

The complete system of equations describing the behavior of the vehicle consists of the vector equation of motion (1.5), the weight relations (1.1)—(1.3), and the control characteristic of the engine (1.4).

The trajectory of the vehicle must usually start at a given point of the phase space (i.e., with given coordinates and velocity components) and reach another point in a fixed time. This requirement may not be satisfied uniquely (owing to the presence of free control functions u_j, $\mathbf{l}$).

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* Formula (1.3) is called the weight formula of the engine, and (1.4) its control characteristic. The form of these relations depends on the type of engine. The choice of a system of independent control parameters depends on the restrictions characteristic of the given type of engine: if explicit restrictions are imposed on any control parameter, it is appropriate to retard it as independent.

From the whole set of trajectories connecting two given points in the phase space $\{\mathbf{r}_0, \dot{\mathbf{r}}_0\}$ *and* $\{\mathbf{r}_1, \dot{\mathbf{r}}_1\}$ *in a fixed time* T, *it is required to select a trajectory which corresponds to motion of the vehicle with maximum payload* G_p *for a fixed launching weight* $G_0 = G_{m0} + G_{\text{ж}0} + G_p$.

This is the variational problem of optimal controls of the engine and optimal relations between the weight components. The control functions are the thrust direction vector and the engine parameters, which determine the thrust and the weight G.

Another, equivalent, variational formulation is: *It is required to determine a trajectory which connects two given points of the phase space in minimum time for given launching weight and payload.* The second formulation may be more convenient in cases when there is no certainty as regards the feasibility of the maneuver in the time range under consideration, from the power aspect.

These two basic variational formulations may vary according to the specific character of the problem, changing the criterion function and the boundary conditions. The latter correspond to the type of space maneuver: inter-orbital transfer, escape from the gravitational field of a planet, evolutions in the neighborhood of a planet, and so on. In such cases the variational formulations do not impose all the boundary conditions on the coordinates and velocities — only part or combination of the coordinates and velocities are given. The free boundary conditions are then determined from considerations of the optimality of the trajectory.

2. Mechanics of solar-sail flight

The basic unit of a propulsion system with a solar sail is a mirror-surface producing thrust by reflection of solar light. The first serious investigation of this problem was made by F. A. Tsander /2/. He showed that realization of interplanetary flights with the aid of solar pressure is possible in principle.

The formulation of the optimization problem for a solar sail requires, according to the general procedure described in Section 1, the specification of relations (1.3) and (1.4). Here the mass flow is zero ($q = 0$, $G_m \equiv 0$), and the thrust produced by a flat, ideally reflecting, sail of area S at a distance R from the sun is

$$\mathbf{P} = p_0 (R_0 / R)^2 S (\mathbf{n} \cdot \mathbf{e})^2 \mathbf{n} \quad (p_0 = 0.944 \cdot 10^{-6}\ \text{kg/m}^2,\ R_0 = 0.1495 \cdot 10^{12}\ \text{m}). \tag{2.1}$$

The only control parameter u_l of this simple propulsion system with constant area S is the angle between the unit vector $\mathbf{n}$ of the normal to the shaded side of the sail and the unit vector $\mathbf{e}$ of the direction of the solar rays. Variation of this angle (in the range $[-{}^1/_2\pi, +{}^1/_2\pi]$) changes both the direction and the magnitude of the thrust.

The weight $G_{\text{ж}}$ of the propulsion unit is determined principally by the weight G_s of the sail itself

$$G_{\text{ж}} \approx G_s = g\rho S\delta, \tag{2.2}$$

where ρ is the mass density of the sail material ($\rho \sim 1 - 2\,\text{g/cm}^3$), δ is the thickness of the sail ($\delta \sim 2 \cdot 10^{-3} - 10^{-2}$ mm).

Using the kinematic parameter

$$a_0 = p_0 S\,(G_p/g + \rho S\delta)^{-1} \leqslant p_0/\rho\delta \tag{2.3}$$

of the acceleration imparted by a sail perpendicular to the solar rays ($\mathbf{n} = \mathbf{e}$) to the cosmic device at a distance R_0 from the sun, the weight components can essentially be eliminated from the optimization problem

$$G_p/G_0 = 1 - a_0(\rho\delta/p_0). \tag{2.4}$$

The mass $\rho\delta$ of unit sail area depends on the quality of the construction and should be the smallest possible. Thus, the relative payload may be regarded as a function of the acceleration a_0.

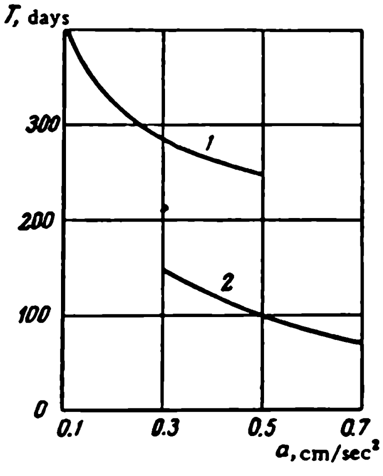
The equation of motion (1.5) can also be expressed in terms of a_0:

$$\ddot{\mathbf{r}} = a_0(R_0/R)^2(\mathbf{n}\cdot\mathbf{e})^2\,\mathbf{n} + \mathbf{R}. \tag{2.5}$$

The optimization problem may then be formulated as follows: *Find a control law of the sail orientation* $\mathbf{n}(t)$ $(|\mathbf{n}(t)| \equiv 1)$ *and a trajectory* $\mathbf{r}(t)$ *(2.5) to ensure minimum* a_0 *(i.e., maximum* G_p/G_0*, by (2.4)) for fixed time T or (by virtue of their mutual reciprocity) minimum T for fixed acceleration* a_0*.* The first formulation is less convenient, since the resulting acceleration a_0 may turn out to be greater $a_{0\max} = p_0/\rho\delta$ (corresponding to $G_p < 0$, see (2.3)).

Interorbital transfer. Most of the current literature on the solar sail investigates plane motion in the central gravitational field of the sun when the angle between the sail and the solar rays is constant. A number of exact and approximate solutions of equation (2.5) have been obtained and the optimum sail angle for minimum interorbital transfer time has been found. These analytic solutions have an essential shortcoming: the boundary conditions for the coordinates and velocities at both ends of the trajectory cannot be satisfied.

The variational problem of an optimum program for the sail angle is solved numerically for given boundary conditions for interorbital transfer in /3, 4/. The first of these papers uses the gradient method of control optimization, while in the second the problem is reduced to a boundary-value problem for a system of differential equations (by means of the maximum principle), and solved by Newton's method. Figure 1 (curve 1 "Earth—Mars" /4/) gives the dependence of minimum transfer time between circular coplanar orbits of Earth and Mars on the initial acceleration (2.3).



1

The escape of a solar-sail vehicle from the gravitational field of a planet is considered in /5, 6/ under the following assumptions: the vehicle is "parked" in an initial circular orbit, the gravitational field is central, the flux of solar light is plane-parallel with constant intensity and direction.

These papers do not study the variational problem of an optimum sail-orientation program. The paper /5/ considers rotation of the sail with angular velocity equal to half the angular velocity of revolution of the vehicle around the planet, the initial orbit and escape trajectory being in the plane of the ecliptic. In /6/ the angle of the sail is chosen from the local-optimality condition of maximum rate of energy increment; the plane of the initial orbit is perpendicular to that of the ecliptic, in order to eliminate shaded sections of the trajectory.

A numerical solution of the Cauchy problem for the equations of motion (2.5) in dimensionless form, with the corresponding sail-orientation programs, has shown that the time needed to reach the escape point by the scheme of /6/ is somewhat shorter than in the case of the two-dimensional trajectory /5/. However, this is not a final indication that acceleration in a plane perpendicular to the ecliptic is preferable, since both papers study nonoptimum sail-orientation programs. A comparison with inter-orbital transfer time shows that a considerable part of the latter is occupied by the time of escape from the gravitational field of a planet (compare, in Figure 1, the curve "Earth—Mars" with curve 2 "escape from the Earth's field" /5/).

3. Mechanics of power-limited flight

Power limited propulsion systems consist of a power generator and propellant unit. The latter transforms the energy produced by the generator into the kinetic energy of the directed motion of the jet which produces the thrust. The presence of a separate generator of limited power determines the basic features and the designations of this category of engines.

Examples of power-limited propulsion systems are electrical jet engines. Power generators mentioned in the literature are, for example, solar collectors, nuclear and isotope reactors; the propellant units may be electrical boosters of various types: electric-arc, electrostatic and electrodynamic.

Formulation of the optimization problem. The basic parameters of power-limited propulsion systems are: q — the mass flow rate of the propellant, V — the exhaust velocity, P — the thrust, N — the jet power. A characteristic feature of these systems is their controllability: the parameters q and V may be varied within wide limits. It is also possible to control the power supply to the engine. If the distribution of the parameters over the cross-section of the jet emerging from the propellant unit is uniform, we have the following two simple relations for mass flow velocity, thrust and jet power

$$P = qV, \quad N = \frac{1}{2} qV^2. \tag{3.1}$$

Thus, only two of these parameters may be considered independent.

The jet power N is always limited by the maximum power N_0 of the generator:

$$0 \leqslant N \leqslant \eta N_0, \tag{3.2}$$

where η is the coefficient of conversion of the generator power into jet power (the efficiency of the propellant unit). In an ideal propellant unit all the input power is transformed into jet power ($\eta = 1$). In actual units losses occur ($\eta < 1$), the efficiency depending on the engine parameters, for example, on the exhaust velocity V. For actual, in contrast to ideal propellant units, restrictions may be imposed on the exhaust velocity or mass flow, apart from those on the power (3.2).

Corresponding to the restrictions imposed, N and V or N and q are taken as the two control parameters. Then

$$P = 2N/V, \quad q = 2N/V^2 \tag{3.3}$$

or

$$P = \sqrt{2Nq}, \quad q = q. \tag{3.4}$$

For an ideal propellant unit (when V and q are unrestricted and $\eta = 1$) it is convenient to represent the thrust and the mass flow in terms of the power N and the kinematic characteristic a—the jet acceleration:

$$P = aG/g, \quad q = a^2G^2/2Ng^2. \tag{3.5}$$

The weight $G_{\text{к}}$ of a power-limited engine is determined mainly by the weight G_N of the generator, which is proportional to the maximum power /7, 8/:

$$G_{\text{к}} \approx G_N = \alpha N_0. \tag{3.6}$$

The specific gravity α of the power generator for large values of N_0 ($N_0 \geqslant 10^3$ kw) is almost constant. The values discussed in the literature are $\alpha \sim 1 - 10$ kg/kw.

Corresponding to the three combinations of the control functions (3.3)—(3.5), the mass flow equation (1.2) and the vector equation of motion (1.5) can be represented in three forms

$$\begin{aligned}
&\dot{G}_{\text{m}} = -g\frac{2N}{V^2}, \quad \ddot{\mathbf{r}} = \frac{2Ng}{V(G_{\text{m}} + G_N + C_{\text{п}})}\mathbf{l} + \mathbf{R};\\
&\dot{G}_{\text{m}} = -gq, \quad \ddot{\mathbf{r}} = \frac{\sqrt{2Nq}\,g}{G_{\text{m}} + G_N + G_{\text{п}}}\mathbf{l} + \mathbf{R};\\
&\dot{G}_{\text{m}} = -\frac{a^2(G_{\text{m}} + G_N + G_{\text{n}})^2}{2Ng}, \quad \ddot{\mathbf{r}} = a\mathbf{l} + \mathbf{R}.
\end{aligned} \tag{3.7}$$

The complete system of control functions comprises the vector-function $\mathbf{l}(t)$ ($|\mathbf{l}| \equiv 1$) which determines the orientation law of the thrust, the function $N(t)$—the jet power ($0 \leqslant N(t) \leqslant \eta N_0$), the function $G_N(t)$—the weight of the power generator (the latter may either remain constant along the trajectory or decrease ($dG_N/dt \leqslant 0$), and one of the functions $V(t)$, $q(t)$, $a(t)$, depending on the form of equations (3.7).

The variational problem for a power-limited engine is formulated as follows: *It is required to determine optimum controls* $\mathbf{l}(t)$, $G_N(t)$, $N(t)$, *and* $a(t)$ *(or* $V(t)$, *or* $q(t)$*) which guarantee delivery of maximum payload in a given maneuver, for given launching weight* G_0, *in fixed time* T *(or given payload* G_{p} *in minimum time* T*).*

The optimal power control is found to be extremal in all cases ($N(t) = \eta N_0$, or 0—maximum power utilization on the active sections). The choice of an optimum law for $G_N(t)$ (where decrease in the weight of the power generator is permitted) or an optimum value for G_N (where the weight of the generator is assumed constant) (the weight part of problem), cannot in general be separated from the choice of the other controls. However, for an ideally controllable engine (or under direct restrictions on the jet acceleration) the weight part of the problem can be solved independently of the choice of the control $\mathbf{a}(t) = a\mathbf{i}$ which determines the trajectory.

Ideally controllable propulsion system — optimum weight relations, optimum power control.

If the weight of the power source is constant, the fifth equation of (3.7), the mass flow equation, can be integrated by quadrature, giving the payload weight in the form /9, 10/

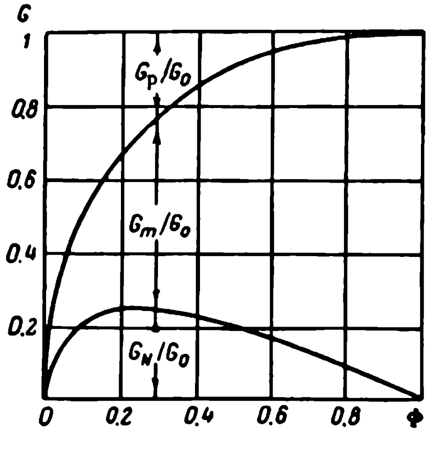
$$G_p/G_0 = \left(1 + \frac{G_0}{2g}\int_0^T \frac{a^2}{N}dt\right)^{-1} - G_N/G_0. \tag{3.8}$$

The control function $N(t)$ appears only in the integrand of (3.8) and not in the equation of motion (the sixth equation of (3.7)). If the power has a constant upper limit $N_0 = \text{const}$, which is the case here, the optimum law for $N(t)$, delivering maximum G_p (corresponding to a minimum of the integral in (3.8)) is determined independently of the other control functions /9/: $N(t) \equiv N_0$.

Taking N outside the integral in (3.8), the integral

$$J = \int_0^T a^2 dt \tag{3.9}$$

remaining in the expression for the payload is independent of the power N_0 and the weights G_N and G_0, and can be calculated if the trajectory (t) is known. This integral represents the trajectory part of the problem in the payload formula. The smaller its value, the greater the payload for fixed G_0, N_0 and G_N. Minimization of J is the trajectory part of the general problem; selection of optimum G_N is the weight part of the general problem.



2

If the weight G_N of the power generator is a linear function of the power N_0, then, for a fixed value of the functional J, $G_p(G_N)$ is a maximum for

$$G_N/G_0=\sqrt{\Phi}-\Phi,\ (\Phi=(\alpha/2g)J) \tag{3.10}$$

and then /9, 10/

$$G_p/G_0=(1-\sqrt{\Phi})^2. \tag{3.11}$$

The optimal weight relations $G_p(\Phi)$, $G_m(\Phi)$ and $G_N(\Phi)$ are represented in Figure 2. Note that for constant weight of the power generator, a maneuver for which $J > 2g/\alpha$ ($G_p=0$ for $\Phi=1$) cannot be performed.

If the relations $G_N(N_0)$ is nonlinear, the optimal weight relations cannot be expressed explicitly. In this general case the equation for the optimum power has the form /11/

$$\frac{dG_N(N_0)}{dN_0}\left(\frac{N_0}{G_0}\frac{1}{\sqrt{J/2g}}+\sqrt{J/2g}\right)^2=1. \tag{3.12}$$

The weight of a power-limited propulsion system amounts, as Figure 2 shows, to a considerable part of the launching weight of the vehicle. If the payload is small, the weight of the engine predominates toward the end of the motion, when most of the propellant has been consumed. It may therefore be expedient to jettison sections of the propulsion unit during the motion.

For heat engines the problem of optimum staging is essentially that of optimum jettisoning of the tanks, since heat engines have low specific gravity. For power-limited engines the problem acquires a new formulation— to arrange for optimum decrease in the weight generator /12/.

Decrease in G_N implies a proportional decrease in the maximum power $N_0(t)=G_N(t)/\alpha$ ($\alpha=\mathrm{const}$); maximum power is used everywhere $N(t)=N_0(t)$.

For n-stage power decrease, the maximum payload is /1, 12/:

$$G_p/G_0=(1-\sqrt{\Phi_1})^2\sum_{i=2}^{n}\left(\frac{1-\Phi_i}{1+\Phi_i}\right)^2, \tag{3.13}$$

where the sum $\Sigma\Phi_i=\Phi$ is given, while the optimum relation between the Φ_i (the distribution over the sections) can be obtained by differentiating (3.13) with respect to Φ_i ($i=1,\ldots,n$). The trajectory part of the problem again reduces to minimizing the functional (3.9).

Figure 3 compares the payload for constant weight of the power generator ($n=1$) and for stepwise ($n=2$) decrease in the weight of the power generator. By stepwise decrease in G_N the maximum value of the functional (3.9) may be increased to $J=2n/g$ ($G_p=0$ for $\Phi=n$).

For an infinite number of sections ($n\to\infty$), we have the limit case— continuous decrease in the weight of the power generator /13, 14/. Replacing N by a new control function $\nu=N/N_0=\alpha N/G_N$, $0\leqslant\nu(t)\leqslant 1$, we rewrite the third pair of equations of (3.7) as

$$\dot{G}_m=-\frac{(G_m+G_N+G_p)^2}{G_N\nu}\frac{\alpha}{2g}a^2,\ \dot{\mathbf{r}}=\dot{\mathbf{v}},\ \dot{\mathbf{v}}=a\mathbf{l}+\mathbf{R}. \tag{3.14}$$

To determine the optimum control functions $G_N(t)$ and $\nu(t)$, we formulate a variational problem for the minimum flight time between two fixed points of the phase space for given G_0 and G_p.

The boundary conditions for the weights are:

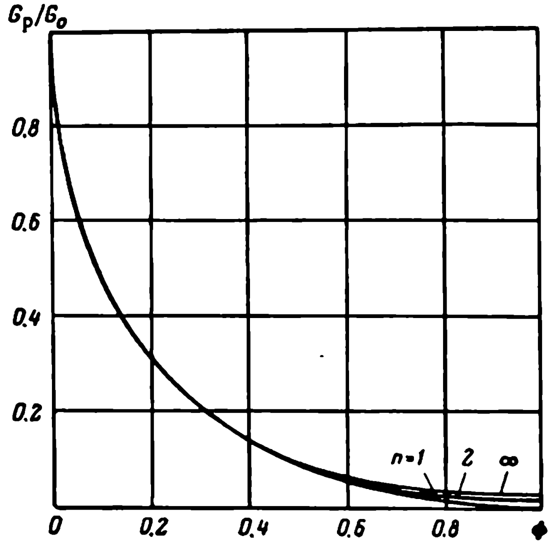
$$G_N(0)+G_m(0)=G_0-G_p,\ G_m(T)=0. \qquad (3.15)$$

The characteristic feature of this problem is that the intial value of the control function G_N is not arbitrary, but occurs in the first of the boundary conditions (3.15). Thus the standard algorithm of variational calculus cannot be used.

Using the maximum principles the optimality of the controls $\nu(t)=1$, $G_N(t)=G_m(t)+G_p$ or $G_N(t)=\mathrm{const}$, is established. Therefore the problem splits into two parts.

1. For a given ratio G_p/G_0, to find the optimal control $G_N(t)$, consisting of the sections $G_N=G_m+G_p$ and $G_N=\mathrm{const}$, which satisfies the initial condition (3.15) and maximizes the functional

$$\Phi=-\int_{G_{m0}}^{0}\frac{G_N dG_m}{(G_m+G_N+G_p)^2}. \qquad (3.16)$$



a

2. For a given value of Φ to find the optimal controls $a(t)$ and $l(t)$ guaranteeing minimum flight time between two fixed points of the phase space.*

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* Hence, for fixed controls $a(t)$, e.g., for $a=\mathrm{const}$/15/, the problem is a special case of this formulation.

The first part deals with the weight, the second with the trajectory. Solution of the first part of the problem yields

$$G_N = \mathrm{const} = (\sqrt{G_p/G_0} - G_p/G_0)\,G_0; \quad \text{for } 0.25 \leqslant G_p/G_0 \leqslant 1$$

$$G_N = \begin{cases} 0.25G_0 & \text{for } 0.25G_0 - G_p \leqslant G_m \leqslant 0.75G_0 - G_p \\ G_m + G_p & \text{for } 0 \leqslant G_m \leqslant 0.25G_0 - G_p. \end{cases} \quad \text{for } 0 \leqslant G_p/G_0 \leqslant 0.25$$

The function $\Phi\,(G_p)$ is (see curve $n \simeq \infty$ in Figure 3):

$$\Phi = \frac{1}{4}\,[1 - \ln(4G_p/G_0)] \quad \text{or} \quad G_p/G_0 = \frac{1}{4}\exp(1 - 4\Phi) \qquad (3.17)$$

The jettisoned sections of the power generator, by an idea of Tsander, can be partially utilized in the propellant unit. Thus we have the problem of optimum utilization of these sections /16, 11/.

Only part of the mass of the jettisoned sections may be converted into propellant: $\gamma\,(t) \leqslant \gamma_{\max} = \mathrm{const} \leqslant 1$ (the parameter $\gamma_{\max}$ is assumed given). The above formulation of the variational problem is maintained, on the whole. The only addition is a new control function $\gamma\,(t)$. The system of equations determining the behavior of the vehicle is expressed in terms of the mass flow (see the second pair of equations of (3.7)):

$$G_m = -gq, \quad \dot{G}_N = -gq_N, \quad \dot{\mathbf{r}} = \mathbf{v},$$

$$\dot{\mathbf{v}} = \sqrt{(2/\alpha)\,G_N\nu\,(q + \gamma q_N)}\,(G_m + G_N + G_p)^{-1}\,\mathbf{l} + \mathbf{R}.$$

Here G_N appears not as a control function, but as a phase coordinate. As a control function for the variation in the weight of the power generator the mass flow $q_N\,(t) > 0$ is introduced. The boundary conditions remain as before (3.15).

The maximum condition for the Hamiltonian function of the problem implies that here too the jet power must be maximal ($\nu\,(t) = 1$). The control $\gamma\,(t)$ is also maximal, $\gamma\,(t) = \gamma_{\max}$. There are three optimal variation laws for the weights G_N and G_m, which can be expressed in terms of the jet acceleration a instead of the mass flows q and q_N. Integration of the above equations yields the final relation between the weights G_p, $G_N\,(0)$, $G_N\,(T)$, G_0, the parameter $\gamma_{\max}$ and the functional Φ; G_p turns out to be a monotone decreasing function of Φ. Thus, as before, the trajectory and weight parts of the problem may be separated.

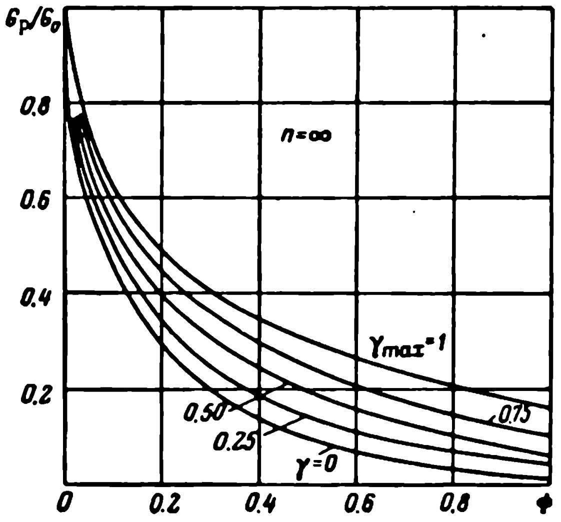
The solution of the latter is shown in Figure 4 as a function $G_p\,(\Phi)$ for various values of $\gamma_{\max}$. Clearly, an active power drop ($\gamma_{\max} > 0$) may yield a significant gain in payload.

Investigation of stepwise variation of the weight of the power generator has shown that the limit payload ($n = \infty$) can be approximated by a finite, comparatively small number of sections n (Figure 5).

Ideally controllable propulsion system — optimal trajectories and programs for the jet acceleration vector. It was shown above that the optimization problem for an ideal system may be divided into two independent problems: 1) to find the optimum ratio between the weight of the power generator and the weight of the propellant, and 2) to find the optimum trajectories $\mathbf{r}\,(t)$ and programs for the jet acceleration vector $\mathbf{a}(t)$, which are related by the differential equation of motion

$$\ddot{\mathbf{r}} = \mathbf{a} + \mathbf{R}(\mathbf{r}, t), \quad (\mathbf{r}(0) = \mathbf{r}_0, \; \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0, \; \mathbf{r}(T) = \mathbf{r}_1, \; \dot{\mathbf{r}}(T) = \dot{\mathbf{r}}_1) \tag{3.18}$$

and minimize the functional (3.9).



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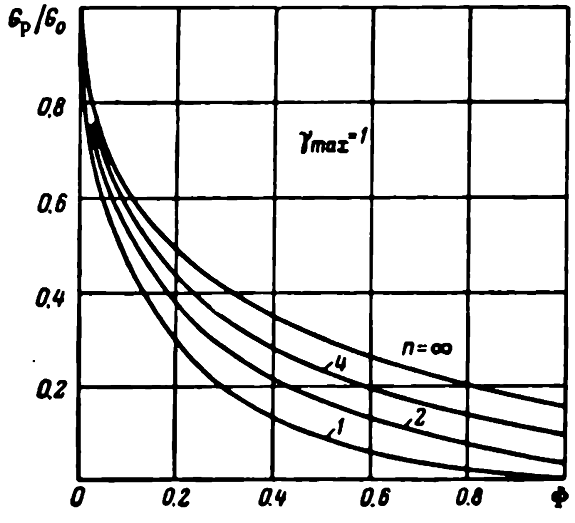


5

Optimal control of $a(t)$ in the ideal case never reaches the limit $a = 0$. Thanks to the absence of limit controls, the laws of optimal motion may be established within the framework of classical variational analysis.

The optimal trajectories and optimum programs for the jet acceleration are found by solving a system consisting of the equations of motion (3.18) and the Euler-Lagrange equations

$$\ddot{\mathbf{a}} = (\partial / \partial \mathbf{r})(\mathbf{a} \cdot \mathbf{R}). \tag{3.19}$$

The system (3.18) and (3.19) is of order 12 in the three-dimensional case. There must also be 12 boundary conditions; one form of these is given above (3.18).

Usually the following three basic trajectory problems are considered: interplanetary transfer, evolutions in the neighborhood of a planet (rotation of the orbital plane, correction of the orbital elements, compensation for a constant force), and reaching a satellite of the planet. The problem most studied at present is that of interplanetary transfer. An interplanetary transfer trajectory has two characteristic sections: motion in the neighborhood of the planet and motion in the field of the sun (concerning the boundary between these sections see /17/). Accordingly, two basic problems for a central gravitational field are formulated: escape from the gravitational field (attaining zero energy) from a given point of the initial orbit in a given time; transfer from a given point of the initial orbit to a given point of the final orbit in a given time. The missing boundary conditions are determined by equating the terms outside the integrals in the first variation of the criterion functional of the problem to zero /17, 18/:

$$J = \int_0^T (\ddot{\mathbf{r}} - \mathbf{R})^2 \, dt. \tag{3.20}$$

If the gravitational acceleration is neglected ($\mathbf{R} = 0$), the system of equations (3.18) and (3.9) is easily integrated

$$\mathbf{a}(t) = \mathbf{b}_0 t + \mathbf{b}_1, \quad \mathbf{r}(t) = \frac{1}{6} \mathbf{b}_0 t^3 + \frac{1}{2} \mathbf{b}_1 t^2 + \mathbf{b}_2 t + \mathbf{b}_3. \tag{3.21}$$

The constant vectors $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_3$ are determined from the boundary conditions. The two basic problems for a central field correspond here to:

1) attaining a given increment Δv of the absolute value of the velocity in time T (in the case of free final coordinates)

$$\mathbf{a}_{\text{opt}}(t) = (\Delta v / T) \mathbf{v}_0 / v_0, \quad J_{\min} = \Delta v^2 / T; \tag{3.22}$$

2) transfer in time T between two stationary points at a distance L from each other:

$$a_{\text{opt}}(t) = (6L / T^2)(1 - 2t / T), \quad J_{\min} = 12L^2 / T^3. \tag{3.23}$$

For plane motion in a central field ($\mathbf{R} = -k\mathbf{r}/r^3$) the system (3.18), (3.19), in polar coordinates (r, φ) can be integrated twice /9, 10, 12/:

$$\dot{r}=v_r,\quad \dot{\varphi}=v_\varphi/r,\quad \dot{v}_r=a_r+v_\varphi^2/r-k/r^2,\quad \dot{v}_\varphi=a_\varphi-v_rv_\varphi/r,$$
$$\dot{a}_r=(1/v_r)\left[{}^1/_2(a_r^2+a_\varphi^2)+a_r(v_\varphi^2/r-k/r^2)-\lambda-\nu v_\varphi/r\right],$$
$$\dot{a}_\varphi=(1/r)(a_\varphi v_r-2a_rv_\varphi+\nu). \tag{3.24}$$

The constants λ and ν are integrals of the system. This is connected with the fact that the equations of motion are independent of the time t and the polar angle φ. These parameters play the role of isoperimetricity constants: the parameter ν corresponds to a given angular displacement $\Delta\varphi$ (for $\nu = 0$ $\Delta\varphi = \Delta\varphi_{\text{opt}}$), and the parameter λ to a given time of motion T (for $\lambda = 0$ $T = T_{\text{opt}}$). The system (3.24) has a singular point—a node—for $v_r = 0$ /12/.

The system of vector equations (3.18), (3.19) has four integrals in the three-dimensional case /18/ and two integrals in the two-dimensional case /18, 19/, analogous to ν and λ. A particular solution of the system (3.18), (3.19) is given in /18/:

$$\mathbf{a}=\beta\mathbf{v}/(1+{}^1/_2\beta t),\quad \beta=\text{const}.$$

A number of approximate solutions of variational problems for a central gravitational field have been obtained.

In the escape problem /12, 20—22/, optimum energy gain occurs for a nearly tangential acceleration direction, due to the smallness of the jet acceleration. If the initial orbit is circular, the trajectory at the start of the motion is a gently sloping spiral, on which the gravitational and centrifugal accelerations are approximately equal $v_\varphi^2/r \approx k/r^2$. This makes it possible to integrate the equations of motion and find the optimal jet acceleration $a_\varphi = \text{const}$, $a_r \approx 0$. In the neighborhood of the escape point this solution is not valid.

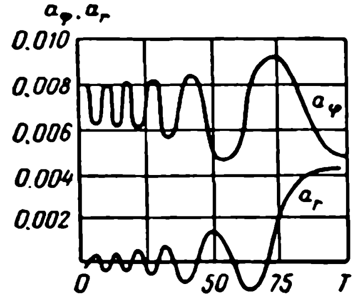
An approximate method for solving the variational equations for the transfer problem is given in /18, 23/. The method is based on the linearization of the equations in the neighborhood of a certain plane Keplerian trajectory, the transporting trajectory, which partially satisfies the boundary conditions (for example, it has the given coordinates at the beginning and end of the motion). The method is applicable when the transfer trajectory does not complete one revolution around the center.

Approximate solutions for optimal three-dimensional motions of a satellite in the neighborhood of a planet are given in /24—27/.

The numerical methods for the problem of optimal trajectories fall into two groups. In the first, the system (3.18), (3.19) (or (3.24)) is integrated, with the corresponding boundary conditions. The boundary-value problem reduces to a Cauchy problem with selection of the missing initial conditions (usually by the Newton method). Alternatively, direct methods may be used to minimize the functional (3.20).

Some typical results for the escape problem /9, 10, 17, 22, 28/ are given in Figures 6—8. All the parameters are dimensionless; the characteristic distance is the radius of the initial circular orbit r_0, the characteristic time is the period of revolution in the initial orbit, divided by 2π—$T_0=\sqrt{r_0^3/k}$.

The absolute value of the jet acceleration is almost constant along the entire escape trajectory. The gain in the functional due to programming of the jet acceleration vector is shown in Figure 8 (the dashed curve corresponds to $a_\varphi = \text{const}$, $a_r = 0$).



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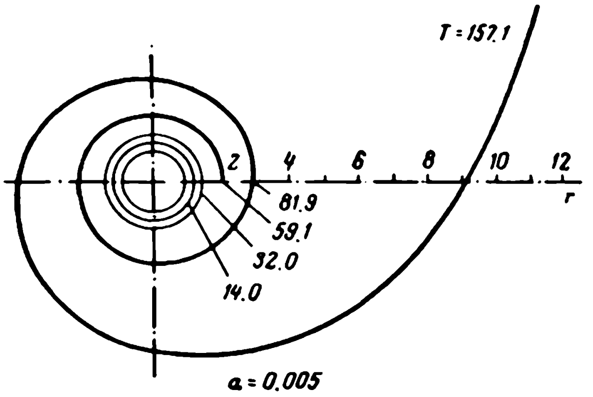

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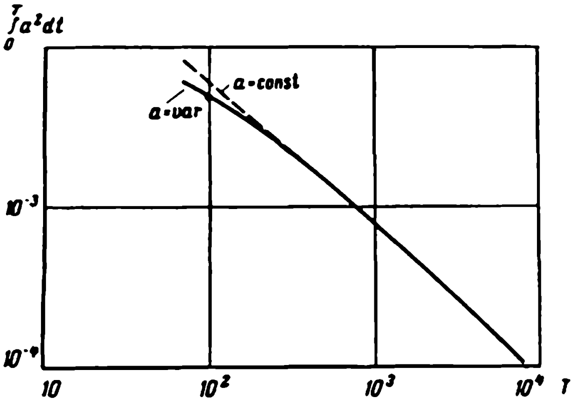

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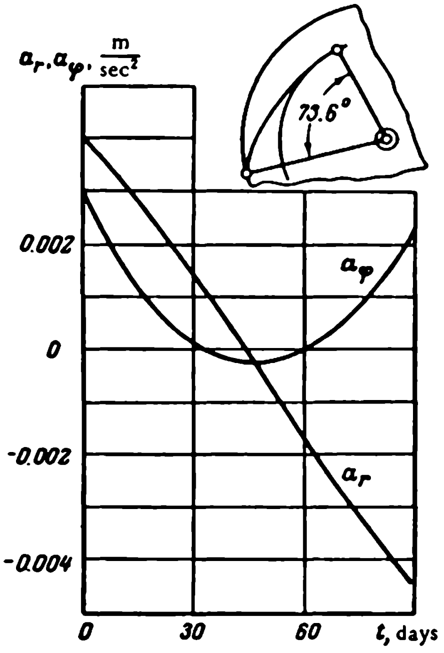

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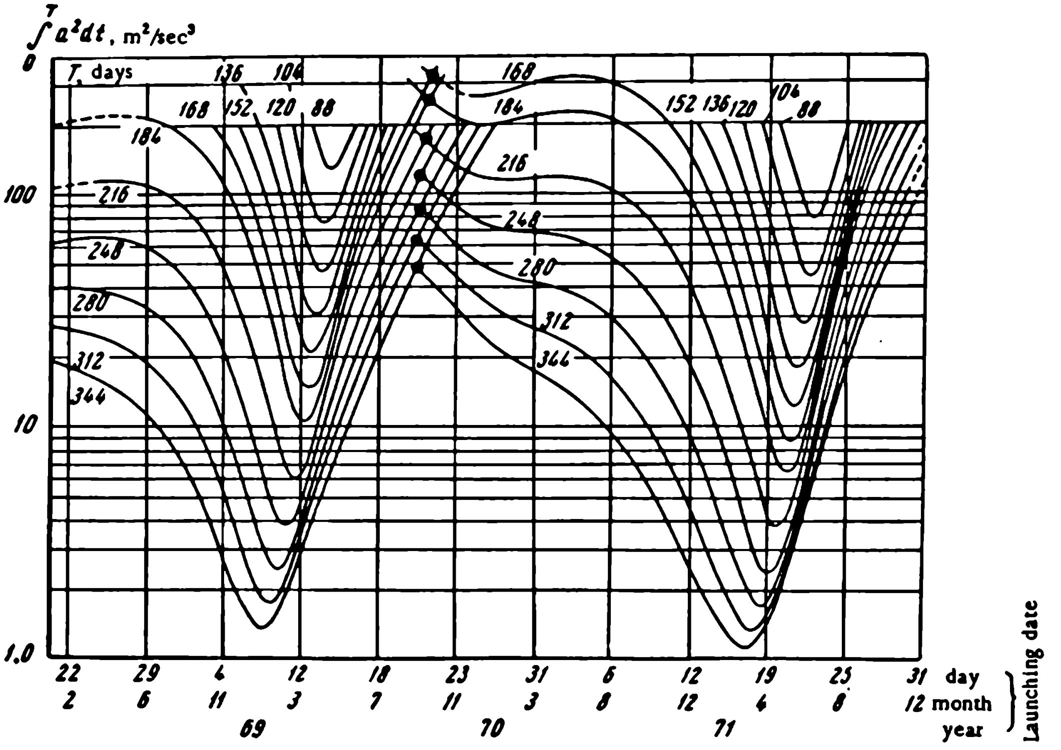
To find optimal transfer trajectories between coplanar orbits, system (3.24) is integrated. If the angular displacement is not given /9, 10, 17, 18, 23/ but taken as optimal, then $\nu = 0$. An example of such an Earth-Mars transfer is shown in Figure 9 (optimal $a_r(t)$, $a_\varphi(t)$ and optimal trajectory). Transfer with optimal angular displacement are possible only for certain positions of the planets in their orbits; the optimal launching dates recur each synodic period.

If a given launching date is added to the given flight time T, or if re-entry is being considered, the angular displacement $\Delta\varphi$ is not optimal, but is chosen to satisfy one of these additional conditions; when this is done $\nu \neq 0$.

Figure 10 /29/ describes the criterion functional for transfer between the circular orbit of Earth and the coplanar elliptical orbit of Mars for various launching dates and flight times. At the points of intersection of the two families of curves of equal flight time the solution of the boundary-value problem is not unique. At these points the time T, the initial and final values of the coordinates and velocities and the functional coincide. Types of trajectories and optimal jet accelerations are distinguished (Figure 11).



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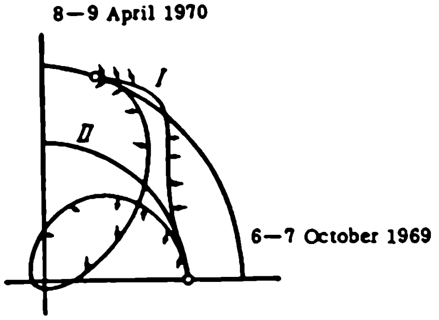
For transfer with re-entry the optimum condition for combination of the trajectories "there" and "back" is the equality of the parameters λ and ν on both trajectories. For transfer between circular orbits this condition is satisfied for "symmetric" trajectories. Symmetric transfer with re-entry is probably optimal for small times. For extended times examples of optimal nonsymmetric flights can be given (Figures 12, 13 from /30, 31/). Data on transfer between circular coplanar orbits of Earth and Mars are presented in /32/, giving the values of the functional and the parameters ν and λ for each pair of values of $\Delta\varphi$ and T. This makes it possible to find the optimal transfer with re-entry.

Estimates of the influence of noncoplanar trajectories /18, 30, 31/ and noncentral gravitational fields /17/ on the integral transfer characteristics have been made. In /10/ the problem of optimum distribution of a given interplanetary transfer time among the different sections is considered. The paper /23/ contains results of systematic calculations of Earth-Mars, Earth-Venus and Earth-Jupiter transfer trajectories with re-entry.



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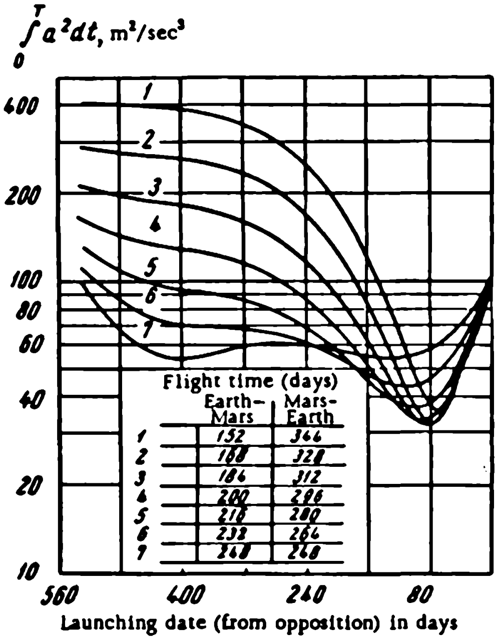


12

Once the trajectory part of the problem has been solved on the basis of the known $a(t)$, all the parameters of an ideally controllable power-limited propulsion unit can be determined: the power N, the thrust $P(t)$, the mass flow $q(t)$, the exhaust velocity $V(t)$; for example, for a single-stage vehicle:

$$N = (1/\alpha)(\sqrt{\Phi_1} - \Phi_1), \quad \left(\Phi_1 = \Phi(T),\ \Phi(t) = (\alpha/2g)\int_0^t a^2\,dt\right);$$

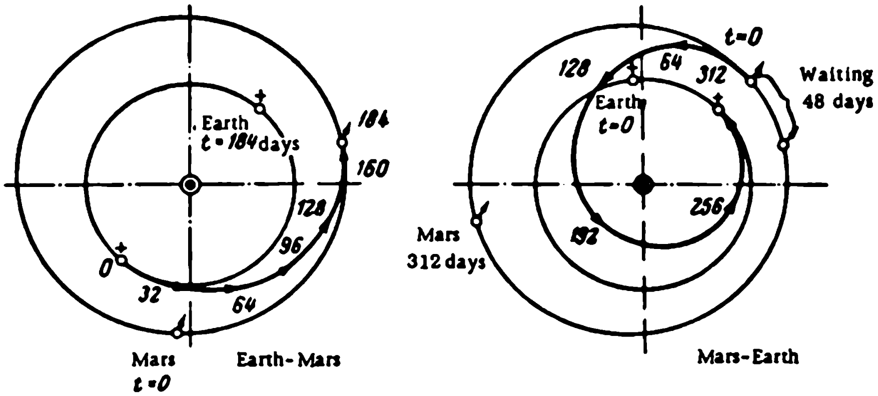
$$P(t) = G(t)\,a(t)/g, \quad (G(t) = G_0(1 + G_0\Phi(t)/\alpha N)^{-1});$$

$$q(t) = G^2(t)\,a^2(t)/2g^2N, \quad V(t) = 2Ng/G(t)\,a(t).$$

The maximum payload G_p can be determined from formulas (3.10), (3.17), and Figures 2—5 for a known final value of the function $\Phi_1 = \Phi(T)$.

The optimization problem as applied to an ideally controllable power-limited propulsion system was briefly described above. The parameters of the propulsion system, the mass flow and exhaust velocity could be varied in this limit case within unlimited ranges, the only condition being the restriction of jet power. The maximum power was considered constant over the whole range of the variation of these parameters. This idealization made it possible to simplify the solution of the problem considerably and to determine the maximum possibilities for this type of propulsion system.

The limited length of this paper does not permit us to describe further investigations of the problem, as applied to real systems. We confine ourselves to an enumeration of the principal trends.



18

1. Uncontrollable propulsion system—optimum weight relations, optimal trajectories and thrust orientation programs /30, 31, 33—35/.

2. Optimal trajectories with constant and limited thrust acceleration /18, 36—39/.

3. Restrictions on the control characteristics of the propulsion system /19, 30, 31, 40/.

4. Restricted operation time of the propulsion system /11, 33/.

5. Restricted number of positions of the controller /41/.

6. Optimum combination of power-limited propulsion systems and restricted exhaust velocity /21, 42/.

7. Reliability considerations in optimization problems for power-limited propulsion systems /14, 43/.

8. Allowance for correction in optimization problems for power-limited propulsion systems /11, 44/.

9. Additional weight components of power-limited engines in the optimization problem /45/.

A description of these problems can be found in the second and third parts of the review /1/. The fourth part of the review describes known analytic solutions of the equations of dynamics, exact and approximate, based on the smallness of the perturbing force, and also numerical methods for design of optimal trajectories and optimal controls.

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V. A. Sarychev

GRAVITY ATTITUDE STABILIZATION OF SATELLITES

1. General principles of the system

An important aspect of the development of space technology is the construction of oriented Earth satellites. Depending on the problems to be solved, the attitude an artificial satellite can be stabilized either actively or passively (i.e., without consumption of energy). Active attitude control systems in projects of long duration (over a year) requiring high precision entail large consumption of power or fuel, and these systems are very heavy and complex. In many cases it is preferable to use passive systems; their merit is especially apparent in long-duration projects where the required precision of attitude stabilization is of order 1°—5°. This is the case, for example, for meteorological satellites, communication satellites, and many scientific research satellites.

Passive stabilization systems may utilize the properties of the magnetic and gravitational fields, the effect of radiation pressure, atmospheric drag, etc. Here we shall consider attitude stabilization of artificial satellites with respect to the trihedron formed by the radius-vector, the transversal, and the binormal to the orbit, called the orbital coordinate system. The stabilization principle is based on the property of the Newtonian force field according to which a body moving in it orients itself in a definite manner. It is well-known /1, 2/ that a body moving in a circular orbit has stable positions of relative equilibrium, in which the major axis of the ellipsoid of inertia of the body coincides with the radius-vector and of the minor axis with the binormal to the orbit; the medium axis is directed along the tangent to the orbit. Examples of natural gravity attitude stabilization of a body with respect to the attracting body are the moon and Mercury.

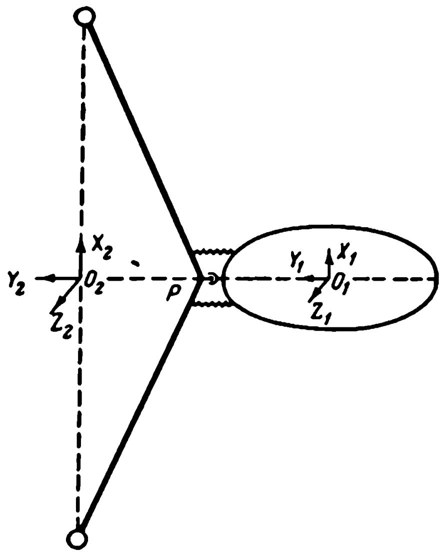
In a conservative Newtonian force field the amplitude of oscillations of a satellite around a stable equilibrium position does not decrease in time; consequently, the precision of the attitude stabilization is determined by the initial values of the angles and angular velocities of the satellite. In stabilization systems for artificial satellites the difficulties associated with the conservativeness of the field are eliminated by introducing dissipative forces. This may be effected by various methods, but in any case the stable positions of relative equilibrium become asymptotically stable.

The nonuniqueness of the stable equilibrium positions imposes restrictions on the initial conditions of the satellite after its separation from the last stage of the carrier rocket. Before a system of gravity attitude stabilization can be introduced, large initial values of the satellite's angles and angular velocities must first be reduced by an active damping system to values which ensure that the satellite cannot pass from one stable equilibrium to another.

2. Problems of design

In /3/ the following fundamental scheme for stabilization and damping of satellites is proposed (Figure 1). A stabilizer is attached to the satellite by means of a three-stage suspension P. The stabilizer may be any body with suitable moments of inertia. In construction it may consist of two booms, rigidly connected to each other, with weights at their ends. Besides the suspension P, which is kinematically equivalent to an ideal spherical hinge, an elastic joint is introduced between the satellite and the stabilizer.

First, using the stabilizer the ellipsoid of inertia of the system as a whole may be reoriented (if necessary). Second, by utilizing the relative motion of the satellite and stabilizer friction may be introduced in the satellite-stabilizer system. The parameters of the stabilizer (boom length, weight, angle between booms) and the elastic joints are chosen so that in a stable equilibrium position of the system in a circular orbit the booms lie in the orbital plane and $O_1X_1 \parallel O_2X_2$, $O_1Y_1 \parallel O_2Y_2$, where $O_1X_1Y_1Z_1$ and $O_2X_2Y_2Z_2$ are the principal central trihedra of the satellite and stabilizer. It is natural that, together with the damper parameters, the parameters of the stabilizer and elastic joints should be chosen to ensure, in addition to the stability conditions, satisfactory dynamics and precision of the stabilization system.



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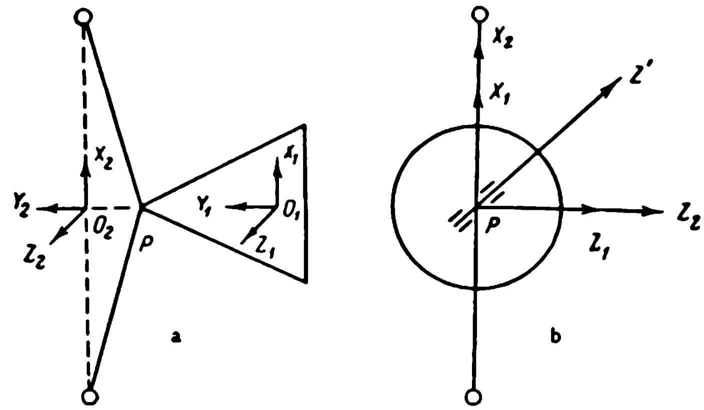
From the constructional viewpoint, the most complicated element of this system is the three-stage suspension. The difficulty is due to the following stringent dynamic requirements: 1) minimum dry friction in the suspension; 2) high precision of its structure; 3) small torques of the elastic joints, of magnitude comparable to that of the gravitational torques. The first two factors determine the precision of stabilization of the satellite, the third largely determines the duration of the transient process.

In /4/, we study whether the suspension may be simplified by reducing the number of degrees of freedom. In deriving the equations of motion of the system in question, it was found that small oscillations in the pitch angle (in the orbital plane) do not depend on the roll and yaw angles, while oscillations in the roll and yaw angles are interrelated and independent of the pitch angle. Clearly, because of the interrelation between the oscillations in the yaw and roll angles, it is sufficient to introduce damping for the pitch and, say, yaw angle. Oscillations in the roll angle must then be damped owing to the dynamic coupling with the oscillations in the yaw angle. This implies simplifying the suspension design by constructing it with two degrees of freedom, eliminating, for example, oscillations of the stabilizer relative to the satellite in the roll angle.

A suspension with a single degree of freedom is also feasible. To this end one must eliminate the symmetry of the scheme of the stabilization system, which leads to the independence of oscillations in the yaw and roll in the orbital plane, and consequently requires damping oscillations in at least two channels. Figure 2 is a diagram of a gravity stabilization system with a suspension having a single degree of freedom. Figure 2a (view from the side of the normal to the orbital plane) illustrates the scheme in the equilibrium position, the plane of the booms coinciding with the orbital plane (the plane of the drawing) and $O_1X_1 \parallel O_2X_2$, $O_1Y_1 \parallel O_2Y_2$. Figure 2b is a front view of the scheme. The stabilizer may oscillate relative to the satellite about the axis PZ', which does not coincide with any principal axis of the satellite. The equations of motion for this gravitational stabilization system are coupled, so that damping of the relative oscillations about the axis PZ' ensures asymptotic stability with respect to all the angular variables.



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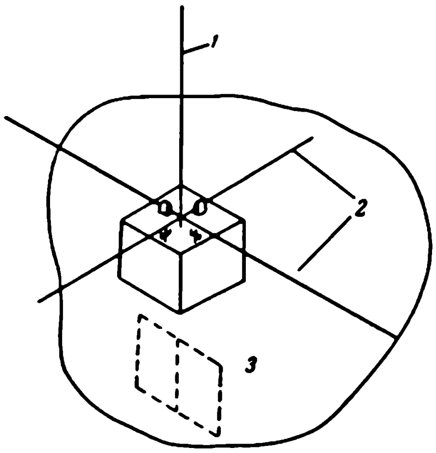

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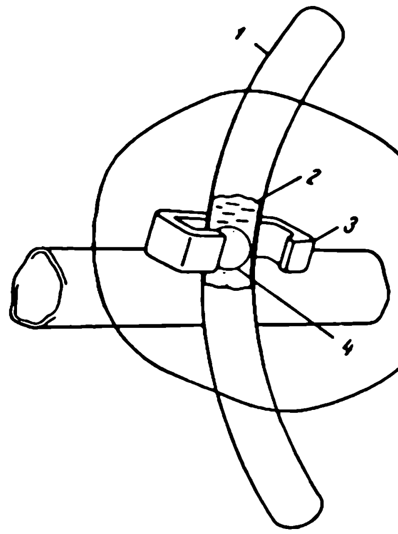

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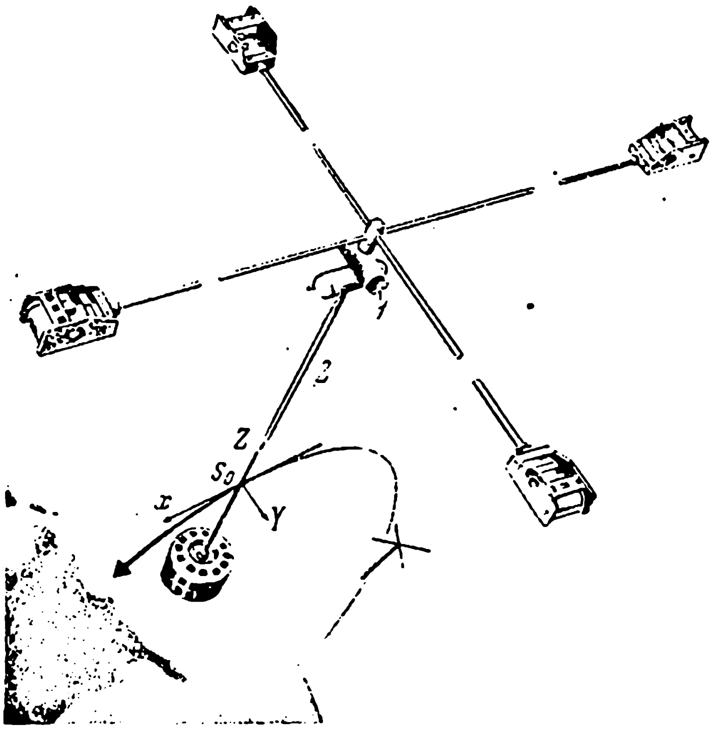
Ideas similar to those of /3, 4/ are expressed in /5, 6, 7, 8/. Paper /5/ gives a detailed description of a suspension scheme coupling the satellite to the stabilizer, and a damping device. The author suggests a stabilizer consisting of three mutually perpendicular tubes of different lengths (Figure 3). The tubes are made of initially contracted metal bands which unroll and assume the required form under the action of elastic forces. The principal (longest) tube 1 is rigidly attached to the satellite and serves to reorient the ellipsoid of inertia of the satellite. The two auxiliary tubes 2, perpendicular to the principal tube, are fastened together by a single-stage suspension and connected to the satellite through a viscous damper, ensuring asymptotic stability of the equilibrium position (3 is the orbital plane).

Figure 4 illustrates a hermetic viscous damper. A part containing a toroidal cavity 1, filled with a viscous fluid 2, is fastened to the body of the satellite, and a magnet 3 to an auxiliary tube. The magnet holds a steel ball 4 between its poles, and so displacement of the magnet relative to the cavity causes displacement of the ball in the viscous fluid and dissipation of energy. This project has been named "Vertistat" and is being developed by the astronautics division of "General Dynamics."



A further investigation of the "Vertistat" gravity attitude stabilization system is given in /6/. The structural details of the self-unwinding coil, the suspension, and the dampers of the auxiliary tubes are considered in detail and the effectiveness of the damping device is experimentally verified. Methods for reducing the deformation of the material of the long tubular booms due to heating by the sun are studied, among them suitable choice of tube material, special coatings, and perforation of the tubes.

An interesting gravity attitude stabilization system for communication satellites has been proposed by "Bell Telephone Laboratories" /7, 8, 10/ (Figure 5). The main difference between this system and that described above consists in the damping device. The oscillations of the satellite-stabilizer system are damped by hysteresis effects in permeable disks moving between small permanent magnets. As stated in these papers, hysteresis damping is preferable to other methods since it is determined by the oscillation amplitudes and is independent of oscillation frequency and temperature variations. Papers /7, 8/ contain a general description of the design of the system and results of tests of the damper device, and consider the influence of various perturbing torques.

3. Dynamics of the system

Papers /9—14/ study the dynamics of gravity attitude stabilization systems, considering the following problems: 1) derivation and analysis of conditions for stability of the equilibrium position; 2) rapidity of damping of natural oscillations; 3) influence of drag; 4) eccentricity oscillations; 5) errors.

A detailed analysis of the three-dimensional motion of a satellite-stabilizer system in circular and elliptic orbits is given in /9, 11, 14/, which adopt the above described scheme of a satellite-stabilizer system with a three-stage suspension. Necessary and sufficient conditions for asymptotic stability of the equilibrium position of the system are derived in general form for motion along a circular orbit in a drag-free medium /9/. The stability conditions result from the construction of a Lyapunov function. The same conditions may be derived, using a much more complicated method, from the Hurwitz inequalities.

The conditions for stability of the equilibrium position of the system are as follows:

$$\begin{gathered}(B_1-A_1)+(B_2-A_2)-M(b_1-b_2)^2>0,\\ (C_1-B_1)+(C_2-B_2)+M(b_1-b_2)^2>0;\end{gathered} \tag{1}$$

$$\begin{gathered}K_2^{(1)}C_1[(B_1-A_1)+(B_2-A_2)-M(b_1-b_2)^2]+\\ +3[(B_1-A_1)(B_2-A_2)-Mb_1^2(B_2-A_2)-Mb_2^2(B_1-A_1)]>0,\\ K_2^{(2)}C_1[(C_1-B_1)+(C_2-B_2)+M(b_1-b_2)^2]+\\ +[(C_1-B_1)(C_2-B_2)+Mb_1^2(C_2-B_2)+Mb_2^2(C_1-B_1)]>0,\\ K_2^{(3)}C_1[(C_1-A_1)+(C_2-A_2)]+4(C_1-A_1)(C_2-A_2)>0;\end{gathered} \tag{2}$$

$$\begin{gathered}K_1^{(1)}>0,\ K_1^{(2)}>0,\ K_1^{(3)}>0,\\ C_1+Mb_1^2>0,\ A_1+Mb_1^2>0,\ B_1>0,\ B_2>0,\\ C_1C_2+Mb_1^2C_2+Mb_2^2C_1>0,\ A_1A_2+Mb_1^2A_2+Mb_2^2A_1>0;\end{gathered} \tag{3}$$

$$\begin{gathered}[(B_1-A_1)-Mb_1(b_1-b_2)][C_2-M(b_1-b_2)b_2]-\\ -[(B_2-A_2)+M(b_1-b_2)b_2][C_1+Mb_1(b_1-b_2)]\neq 0.\end{gathered} \tag{4}$$

Here $|b_1| = O_1P; |b_2| = O_2P; M = M_1M_2/(M_1 + M_2)$; $K_1^{(1)}, K_1^{(2)}, K_1^{(3)}$ are the dimensionless friction coefficients in the damper; $K_2^{(1)}, K_2^{(2)}$ and $K_2^{(3)}$ are the dimensionless coefficients of the elastic forces in the suspension; $M_1, A_1, B_1, C_1, M_2, A_2, B_2$ and C_2 are the masses and principal central moments of satellite and stabilizer, respectively.

The physical meaning of inequalities (1) is that the satellite-stabilizer system should be gravitationally stable in the position of equilibrium, i.e., the axis of the smallest moment of inertia of the system as a whole coincides with the radius-vector, the axis of the largest moment coincides with the binormal to the orbit.

Conditions (2) impose a lower bound on the fixing torques and constitute a strengthening of (1).

The first three conditions of (3) represent the need for friction in the system to ensure asymptotic stability. The remainer of (3) represents restrictions on the design of the satellite and stabilizer, connected with the damping of the initial perturbations with respect to all the angular variables. Condition (4) precludes the particular solution according to which the satellite-stabilizer system performs oscillations as a rigid body without dissipation of energy.

Stability analysis for the gravity attitude stabilization system developed by "Bell Telephone Laboratories" is carried out in /10/, which derives general conditions for asymptotic stability for two-dimensional oscillations. As for the stability of three-dimensional oscillations for given values of the system's parameters, the authors suggest using the usual determinant inequalities of Hurwitz.

The rapidity of damping of the natural oscillations is considered in /9—13/. In /9—11/, the criterion for minimum duration of the transient process is that among the real parts of the roots of the characteristic equation, that having the smallest absolute value should be a maximum. These papers determine the optimal parameters and present diagrams describing the optimal transient process of the system for three-dimensional oscillations in a circular orbit. Papers /12, 13/ deal with the determination of the time optimal parameters of the system for two-dimensional oscillations. In /10/ the problem of optimizing the transient process is solved by numerical integration.

Depending on the design of the satellite-stabilizer system and the altitude of the orbit, one must often allow for atmospheric drag /11/. We shall assume that 1) the atmosphere is completely entrained by Earth's rotation; 2) the orbit is circular; 3) O_1Y_1 and O_2Y_2 are axes of symmetry of satellite and stabilizer, respectively; 4) the effect of the atmosphere on the body reduces to a drag force acting at the center of pressure in a direction opposite to the velocity of the center of mass of the body. The necessary and sufficient conditions for asymptotic stability of the natural oscillations of the system in a circular orbit with allowance for atmospheric drag have the form:

$$\begin{gathered}3\left[(B_1 - A_1) + (B_2 - A_2) - M(b_1 - b_1)^2\right] + C_1(\varkappa_1 + \varkappa_2) > 0, \\ \left[(C_1 - B_1) + (C_2 - B_2) + M(b_1 - b_2)^2\right] + C_1(\varkappa_1 + \varkappa_2) > 0, \\ \left[(C_1 - A_1) + (C_2 - A_2)\right] > 0;\end{gathered} \qquad (5)$$

$$\begin{gathered}
K_2^{(1)} C_1 \{3[(B_1 - A_1) + (B_2 - A_2) - M(b_1 - b_2)^2] + C_1(\varkappa_1 + \varkappa_2)\} + \\
+ 3C_1\varkappa_1[(B_2 - A_2) - Mb_2^2] + 3C_1\varkappa_2[(B_1 - A_1) - Mb_1^2] + C_1^2\varkappa_1\varkappa_2 + \\
+ 9[(B_1 - A_1)(B_2 - A_2) - Mb_1^2(B_2 - A_2) - Mb_2^2(B_1 - A_1)] > 0, \\
K_2^{(2)} C_1 \{((C_1 - B_1) + (C_2 - B_2) + M(b_1 - b_2)^2] + C_1(\varkappa_1 + \varkappa_2)\} + \\
+ C_1\varkappa_1[(C_2 - B_2) + Mb_2^2] + C_1\varkappa_2[(C_1 - B_1) + Mb_1^2] + C_1^2\varkappa_1\varkappa_2 + \\
+ [(C_1 - B_1)(C_2 - B_2) + Mb_1^2(C_2 - B_2) + Mb_2^2(C_1 - B_1)] > 0, \\
K_2^{(3)} C_1[(C_1 - A_1) + (C_2 - A_2)] + 4(C_1 - A_1)(C_2 - A_2) > 0;
\end{gathered} \tag{6}$$

$$\begin{gathered}
\{C_1\varkappa_1 + 3[(B_1 - A_1) - Mb_1(b_1 - b_2)]\}[C_2 - M(b_1 - b_2)b_2] - \\
- \{C_1\varkappa_2 + 3[(B_2 - A_2) + M(b_1 - b_2)b_2]\}[C_1 + Mb_1(b_1 - b_2)] \neq 0.
\end{gathered} \tag{7}$$

Here

$$\varkappa_1 = \frac{-Q_1 b_3 - (\overline{M}_1 Q_2 - \overline{M}_2 Q_1) b_1}{C_1 \omega_0^2}; \quad \varkappa_2 = \frac{-Q_2 b_4 + (\overline{M}_1 Q_2 - \overline{M}_2 Q_2) b_2}{C_1 \omega_0^2};$$

$\overline{M}_1 = \frac{M}{M_1 + M_2}$; $\overline{M}_2 = \frac{M_1}{M_1 + M_2}$; Q_1 and Q_2 are the drag forces acting on satellite and stabilizer; $(0, b_3, 0)$ and $(0, b_4, 0)$ are the coordinates of the centers of pressure of satellite and stabilizer with respect to the trihedra $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$; ω_0 is the angular velocity of revolution of the satellite in orbit. Conditions (3) are independent of the atmospheric drag. From (5) and (6) one easily sees that while stability of the system with respect to roll is due, as before, to the gravitational torque, stability with respect to pitch and yaw can be ensured by the aerodynamic torque, if $\varkappa_1$ and $\varkappa_2$ are taken sufficiently large and positive.

Because of the entrainment of the atmosphere by Earth's rotation, forced oscillations with frequency ω_0 form in the yaw and roll angles. The amplitude of these oscillations is proportional to the parameter

$$\varkappa = \frac{\Omega}{\omega_0} \sin i$$

(where Ω is the angular velocity of Earth's rotation and i is the inclination of the orbit) and decreases with the altitude. The maximum amplitude may be as much as 2—3 degrees.

It should be noted that, in principle, the influence of atmospheric drag on the oscillations can be eliminated. To this end it is sufficient to make $\varkappa_1 = \varkappa_2 = 0$. The atmospheric drag then affects only the translational motion of the system and not the oscillatory motion.

When inequalities (1)—(4) are satisfied in a circular orbit in a drag-free medium, the oscillations die out in time and the satellite-stabilizer system attains a position of stable equilibrium. In an elliptical orbit there is no position of equilibrium. The system performs forced (eccentricity) oscillations resulting from the nonuniform rotation of the orbital coordinate system. The eccentricity oscillations occur in the orbital plane. The oscillation amplitude R_1 and R_2, of satellite and stabilizer respectively, are given by the following expressions:

$$R_1^2 = 4e^2 \frac{a^2}{c^2} \frac{\left(K_2^{(1)} + \frac{b}{a}\right)^2 + (K_1^{(1)})^2}{\left(K_2^{(1)} + \frac{d}{c}\right)^2 + (K_1^{(1)})^2}, \quad R_2^2 = 4e^2 \frac{a^2}{c^2} \frac{\left(K_2^{(1)} + \frac{\bar{b}}{a}\right)^2 + (K_1^{(1)})^2}{\left(K_2^{(1)} + \frac{d}{c}\right)^2 + (K_1^{(1)})^2}, \tag{8}$$

where e is the eccentricity of the orbit, and

$$\begin{gathered}
a = (C_1 + C_2) + M(b_1 - b_2)^2, \\
b = [C_1 + Mb_1(b_1 - b_2))][3(B_2 - A_2) - C_2] - 4Mb_2(b_1C_2 + b_2C_1), \\
c = [3(B_1 - A_1) - C_1] + [3(B_2 - A_2) - C_2] - 4M(b_1 - b_2)^2, \\
d = [3(B_1 - A_1) - C_1][3(B_2 - A_2) - C_2] - \\
- 4Mb_1^2[3(B_2 - A_2) - C_2] - 4Mb_2^2[3(B_1 - A_1) - C_1], \\
\bar{b} = [3(B_1 - A_1) - C_1][C_2 - M(b_1 - b_2)b_2] - 4Mb_1(b_1C_2 + b_2C_1).
\end{gathered}$$

The amplitude of the eccentricity oscillations is proportional to the eccentricity of the orbit and depends on the inertia parameters of the system and the coefficients of friction and elasticity. It follows from (8) that if $K_1^{(1)} = 0$ and $K_2^{(1)} + b/a = 0$, then $R_1 = 0$. Thus, the stabilizer parameters can be chosen so that the amplitude of eccentricity oscillations of the satellite in an elliptic orbit is zero. In this case the stabilizer plays the role of a dynamic damper.

In /3, 9, 11/ the basic case, in which the center of the suspension coincides with the point of intersection of the principal dynamic axes O_1P and O_2P of satellite and stabilizer, is considered. In a circular orbit it corresponds to exact tracking following of the orbital coordinate system. However, this case cannot be realized in practice, due to errors in the construction of the system, namely: 1) the error in determination of the center of mass; 2) the error in determination of the principal dynamic axes; 3) the error in the zero reading of the elastic torques.

These errors constitute random influences on the parameters of the system. From the viewpoint of stabilization precision, the most interesting deviations of the parameters from the rated values are those leading to an inhomogeneous system of differential equations of motion. Any small deviations of the parameters which do not modify the homogeneity of the equations cause only small variations of the transient process and the characteristics of steady-state motion in an elliptical orbit.

By analysis of the precision of the stabilization system, the stabilization precision may be related to the errors in determination of the principal dynamic axes and of the position of the center of mass, and to the errors in the zero reading of the elastic torques. Analysis has shown that in weakly elliptical orbits static errors predominate. Stabilization errors connected with uncertainty in the parameters of the external medium or with neglected external effects have not been considered.

Since the "Vertistat" and "Bell Telephone Laboratories" projects are being developed for comparatively high orbits (including 24-hour orbits), the moments of inertia of the satellite must be increased in order to provide the necessary restoring gravitational torques. In the projects in question this is achieved by spacing masses on long booms (up to 100 m). The booms are tubes fabricated from thermally treated beryllium-copper bands. For systems with long tubular booms the main perturbations are caused by high-frequency elastic oscillations in the booms and nonuniform solar heating of the tube surface. An elementary examination of these problems is contained in /6, 7, 8/.

4. Practical implementation of the projects

The first experimental test of a gravity attitude stabilization system was carried out on the satellite "TRAAC", launched together with the satellite "Transit-IV B" by a single "Thor-Able Star" carrier rocket on 15 November 1961 into an orbit with $h_a = 935$ km, $h_\pi = 850$ km, $i = 32.5°$. The results of this experiment are not known, but they were apparently quite successful, since on 18 December 1962 the first serviceable navigation satellite "Transit-V A" with a gravity attitude stabilization system was launched. This launching was unsuccessful, since the systems intended for receiving commands and information from ground stations failed on 19 December.

In 1963 the satellite "1962 22A" in the same "Transit" series was launched. A detailed description of this satellite and results of the launching are given in /14/. The satellite is shown in Figure 6. Its weight is ~ 80 kg, $h_a = 770$ km, $h_\pi = 720$ km, $i = 90°$, the length of the metal boom is 30 m, the equilibrium length of the spring is 6 m, the weight at the end of the spring is 1.75 kg.

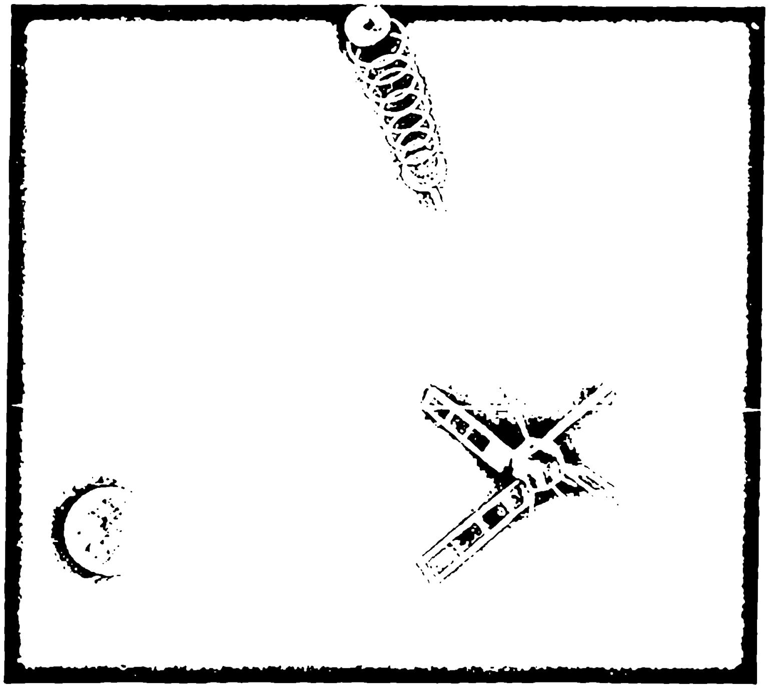
The boom permits elongation of the ellipsoid of inertia of the satellite, and the spring promotes dissipation of energy. The spring is made of a beryllium-copper wire of diameter 0.2 mm. To ensure good damping, the wire is coated with a cadmium layer 0.02 mm thick, and to prevent sublimation of the cadmium in the high vacuum a layer of gold 0.005 mm thick is electrolytically deposited on the cadmium surface.

The boom is a narrow steel or copper-beryllium strip (50 mm wide, 0.05 mm thick), which can be rolled up into a tube by thermal treatment, so that its edges overlap by 180°. This provides the same resistance to bending and rigidity as a seamless tube of the same diameter and wall thickness. Before the satellite is put into orbit the strip of the boom is rolled up, and the weak damping spring is immersed in a block of solid sublimating material — diphenyl.

Before separation from the rocket, the angular velocity of the satellite is ~ 200 rpm. Immediately after separation the velocity of revolution of the satellite is reduced by a "yo-yo" device to ~ 1 rpm; a magnetic damper then reduces the angular velocity to almost zero, and an electromagnet, which orients the axis of the satellite in the direction of the magnetic field (which is close to the local vertical for the selected orbit) is activated.

The boom was extended by command from Earth when the satellite axis formed an angle of $\sim 1°$ with the direction of the magnetic field intensity, and $\sim 12°$ with the local vertical. Complete extension of a 30 m boom took 77 sec. Immediately after extension of the boom the telemetric system showed that the diphenyl blocking device was functioning. Ten hours later, when the diphenyl had evaporated, the damping spring began to free itself. The spring was released completely in 24 hr.

To control the precision of the stabilization system, three-component magnetometers with resolving power + 0.5° and solar direction sensors with resolving power ± 1° were used. To obtain additional information, a flashing lamp with a source powered by solar batteries was used as the mass at the end of the spring. The operating frequency is a little less than one flash per minute. The degree of elongation of the spring may be estimated from the light intensity measured at the satellite. The position of the light source may also be determined within the limits of a solid angle of 5°, 10°, 15° relative to the axis of the satellite.



For an orbiting satellite with average altitude ~ 720 km, the principal perturbing torques are determined by the interaction of the satellite with the magnetic field of the Earth, solar radiation, and atmospheric drag. By calculation, the deviation of the axis of the satellite 22A from the direction of the local vertical due to the perturbing magnetic torque was less than 1°, and that due to atmospheric drag ~ 0.9°. In a satellite with asymmetric surface distribution relative to the center of mass, a perturbing torque is caused by solar radiation pressure. For the satellite 22A this effect yields a deviation of 0.3° from the local vertical. The influence of solar radiation on the stabilization precision is more important; it is due to the nonuniform heating of the long boom, resulting in bending and deviation of the satellite axis from the local vertical by ~ 5°. In addition to the steady constant deviation, when the satellite emerges from the shadow of the Earth into sunlight the boom is given a pulse which may cause oscillations of the boom and weight at their natural frequency. The deviation of the satellite axis from the local vertical due to the ellipticity of the orbit reaches ~ 1.15°.

By processing telemetric data, it was established that all the elements of the gravity attitude stabilization system, including the magnetic devices for reducing the rotational energy, the magnetic orientation devices, the extendable boom, the damping spring, and the attitude control system were functioning satisfactorily. Unexpected high frequency oscillations of the boom were observed, probably caused by the impulsive temperature change in crossing the shadow boundary. This effect leads to stabilization of the satellite with a maximum deviation of approximately 10° from the vertical. The planned 20° precision of orientation of the antenna to the Earth was thus realized.

The bending of the boom due to heat can be considerably reduced by coating it with material which reflects solar energy intensely. Additional coating of the boom strip also reduces bending considerably. Thus, for example, by coating the boom strip with silver and increasing its overlap angle from 90° to 180° the thermal bending is reduced by a factor of 20. This will probably facilitate the solution of the problem of static and dynamic thermal bending, which arose in the satellite "1963 22A", and will permit reduction of the deviation of the satellite axis from the local vertical to 1°.

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* [International Astronautical Federation.]

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SOME PROBLEMS OF ASTRODYNAMICS

The origin and subject of astrodynamics

Rapid progress in the investigation and conquest of space has given birth to a new field of mechanics — astrodynamics, or the mechanics of space flight. The literature dealing with various aspects of the motion of space vehicles is considerable and increasing steadily.

The field is also known as "cosmodynamics", "theory of motion of artificial celestial bodies", "celestial ballistics," and so on. We adopt the term "astrodynamics" as shorter and fairly wide-spread in the literature.

Astrodynamics studies the motion of both unguided and guided space vehicles, develops and uses methods of orbit design, and solves other dynamic problems concerned with the realization of space flight.

Formally, astrodynamics is a part of celestial mechanics as defined by Laplace — the science studying various mechanical motions of celestial bodies due to gravitational forces and others. Astrodynamics may be regarded as a modern branch of celestial mechanics; nevertheless, the term "celestial mechanics" may safely be applied to this science.

Natural and artificial celestial bodies obey the same laws of motion. Astrodynamics therefore uses the methods and results of celestial mechanics extensively. At the same time, the specific features of the subject matter and the approach have led to consideration of new problems and elaboration of new methods outside the scope of traditional celestial mechanics.

Classical celestial mechanics was occupied with the study of the motion of natural celestial bodies; however, as man began to penetrate space it became necessary to design motions, space flights, and orbits, with a view to optimum satisfaction of certain requirements. Typical of these new problems are the control of space vehicles in flight, orbit correction, maneuvering, motion of space vehicles by low thrust operating during a considerable part of the flight.

The classical part of mechanics — the study of the motion of a body about its center of mass — has undergone an interesting development in connection with the rotational and liberational motion of satellites under perturbations. Many other dynamical problems have arisen.

In the sequel we shall briefly consider only a few problems of astrodynamics.

Orbit design

As indicated above, the development of space flight entailed the elaboration of various methods of orbit design. By orbit design we mean selection of space orbits with a view to as complete a solution as possible of the basic problem—flight with the most economical use of technical means.

In a preliminary examination, it is essential to study all the orbits which provide, even only in principle, a solution of the basic flight problem. For example, when planning a flight to the moon it is important to describe all feasible trajectories, to be able to determine the initial conditions necessary for the realization of any specific trajectory, to determine the kinematic and dynamic characteristics of the orbits (flight time, velocity of impact with the moon, required initial energy, observation conditions from given points on Earth, etc.).

In the analysis one may select orbits which are optimal both as regards effective solution of the basic flight problem, and as regards simplicity and economy of realization. These requirements generally conflict, and the final solution is often a compromise, allowing for available technological posssibilities.

One of the most important requirements imposed on an orbit is economy in power consumption for launching and injection into orbit. There are two basic methods for injection: continuous boost and launching from a satellite orbit. The first is technically simpler, but in some cases its realization involves difficulties. The selection of the firing site is inevitably restricted, and the boost of a space vehicle with a given destination may require trajectories which are steeply inclined to the local horizon. This increases the losses due to overcoming gravitational force and reduces the terminal velocity or (for given injection conditions) the weight of the vehicle. The method of continuous boost restricts the range of velocity directions at the beginning of the orbital motion, giving preference not only to orbits with as low an initial velocity as possible, but also to orbits with as small an inclination as possible of the velocity vector to the local horizon at the end of the boost phase.

In the method of launching from a satellite orbit there are no power restrictions on the boost direction. Any direction of the velocity vector is obtainable by suitable timing of the launching into an intermediate satellite orbit (this provides, so to speak, azimuthal sighting, thanks to the rotation of the intermediate orbit together with Earth in its diurnal motion) and suitable timing of the launching from the satellite orbit (this provides, so to speak, sighting with respect to the angle of elevation, since departure from the satellite orbit occurs when the motion in the satellite orbit has the required direction). Boost of a space vehicle both when entering a satellite orbit and when leaving it occurs at minimum angles of inclination to the local horizon and makes maximum use of the capacity of the carrier rocket. Soviet mastery of the method of launching from satellite orbits is an outstanding technical achievement. The use of such a method of injection only restricts the magnitude of the initial velocity in the orbit, permitting injection of heavy vehicles and thus extending the range of possible orbits and facilitating the conditions for their judicious selection.

If the satellite is first to be placed in an intermediate orbit, the most preferable orbits will be those with the lowest initial velocity. In the case of continuous boost it is important that both the velocity and the angle to the local horizon be as small as possible.

Another group of problems connected with the orbit design investigates the precision necessary in the selected nominal orbit and the choice of the correction method. If no corrections occur during the flight, the problem is to determine the deviation limits of the parameters at the end of the boost phase, so that the basic flight problem is solvable if the deviations do not exceed these limits. For example, if the flight destination is the moon, the only admissible deviations of the injection parameters are those for which the orbits pass through the moon. Naturally, the less restrictive the limitations on the region of spread of the injection parameters, the simpler the realization of the flight, the lower the precision required of the equipment effecting the escape, the smaller its weight, and the higher its reliability. Thus, the best orbits for a space flight are those permitting the largest deviations of the injection parameters. This requirement may, and usually does, clash with the power optimality of the orbit; this situation is characteristic in orbit design problems.

It may turn out that the permissible region of initial deviations is excessively small and is not feasible at the present stage of technology. Moreover, our knowledge of celestial-mechanical constants, such as the solar paralax or the orbital elements of planets, may be inadequate, so that even ideal realization of the injection conditions does not guarantee success. In these cases one must use orbit correction during the flight—corrections of the parameters of motion, e.g., by transmitting pulses of suitable magnitude and direction at certain positions in the orbit. Orbit correction can be performed either once or several times during the flight.

For orbit correction the vehicle must have a correcting propulsion unit and a propellant reserve. The additional weight required on board for orbit correction depends on the magnitude of the correcting pulse or, in the case of several corrections, on the total pulse. The magnitude of the correcting pulse depends on the spread of the parameters of motion at the end of the boost phase and will be larger, the larger the spread. In addition, it depends on the position in the orbit at which the correction is carried out. For example, a correction carried out too close to the target may require an excessive velocity change and a large correcting pulse, and, consequently, a considerable additional weight on board the vehicle.

When selecting orbits we must give preference to orbits in which correction is as simple and economical as possible. At the same time the problem arises of optimizing the correction, that is, of selecting the orbit and the correction positions in such a way that the corrections require minimum total pulse and minimum additional weight on board the vehicle.

Solution of the correction problem requires precise determination of the actual parameters of motion during the flight, computation of their deviations from the nominal parameters and of the necessary correction parameters.

Determination of orbit parameters is a classical problem of celestial mechanics. However, for space vehicles its solution involves a number of specific requirements. For example, it is often necessary to determine the orbit parameters as quickly as possible. The computation algorithm, which usually includes an iterative process, should therefore be highly economical and require few iterations, each performable in a short time. The algorithm must also be highly reliable and trouble-free, guaranteeing convergence if the initial approximation is chosen badly.

The precision of the determination of the actual orbit parameters depends on the composition and accuracy of the measured parameters, also on the location of the measured interval in the orbit and its length. As a rule, for given composition and accuracy of the measurements, the more precise the determination of the parameters of the actual motion, (and thus the more reliable and precise the possible orbit correction), the larger the section of the orbit over which the measurement is performed. Excessive prolongation is, however, not desirable, since it may lead to too late a correction and an excessively large correcting pulse.

An early correction may be more economical, but inadequate accuracy in determination of the orbital parameters before it is performed may result in inadequate correction precision and entail a repeated correction.

The above considerations illustrate the complexity and contradictory nature of the problems involved in the design of an orbit measurement and correction system, in other words, in the design of the flight control system. An optimum solution would seem to be a system solving the basic flight problem in the simplest, most reliable, and most economical way (relative to the weight on board). Preference should thus be given to orbits ensuring the most optimal flight control.

In orbit design one must clarify and allow for the diverse conflicting requirements imposed on the orbit, some of which are briefly indicated above; the orbit must then be subjected to a complex analysis and selected with an eye to maximal satisfaction of the formulated requirements. In a thorough flight analysis a great number of alternatives must be calculated. Nevertheless, the computational accuracy required at the initial planning stage is usually not too high. A reasonable solution is therefore to develop and use various approximate techniques for simple, economical and clear (though of limited accuracy) analysis of the relevant requirements, and to aim at a compromise which provides the best solutions of the problem as a whole.

The last planning stage for the alternatives selected requires more accurate calculations, allowing for all factors affecting the flight of a space vehicle. These calculations usually employ numerical integration, using the most accurate values of the constants, in order to obtain precise values of the parameters of flight and injection. Since the refined calculations are often very laborious, the need for effective computational methods is no less pressing here than in the preliminary planning stage. An effective technique should combine the requisite accuracy with speed. In creating the techniques, therefore, maximum use must be made of the known factors. For example, the motion of a space vehicle around Earth in the latter's sphere of activity approximates motion along a conic section with its focus at Earth's center. Motion outside the sphere of activity of Earth approximates heliocentric motion along an unperturbed orbit, and so on. These facts open the way for a perfected technique of refined calculations. There are of course alternative methods.

The methods for investigation of orbits are laregely determined by the character of the flight. One may distinguish between multi-revolution orbits and orbits with small angular distance. The former include the orbits of artificial Earth satellites, the moon, and the planets, which perform a large number of revolutions during their lifetime. The study and design of such orbits involves methods clarifying the evolution of the parameters of the osculating orbit in time, under the influence of perturbing factors such as the noncentrality of the gravitational field, the effect of the atmosphere, perturbations by other celestial bodies, the effect of radiation pressure, etc.

Computation of the evolution process may be treated as a problem of nonlinear oscillations, and extensive use of various averaging methods and asymptotic techniques may lead to simple and effective techniques for both the preliminary and the refined calculation.

Examples of orbits with a small angular distance are flight orbits from Earth to the moon, from Earth to Mars, Venus, and other planets. In the first approximation, these orbits are arcs of conic sections, and no evolution problems arise here. Approximate techniques may conveniently be set up either by neglecting perturbations, or by allowing for them in a fairly crude manner. Thus, a flight orbit to Mars can be considered to consist of portions of conic sections—unperturbed geocentric motion in the activity sphere of Earth, unperturbed heliocentric motion outside the activity spheres of Earth and Mars, and an unperturbed conic section with focus at the center of Mars in the latter's sphere of activity.

The foregoing considerations are only a few of those involved in orbit design.

Investigation of the motion of space vehicles about the center of mass

The motion of an unguided space vehicle about its center of mass is either rotational or librational. In rotational motion the work of the external torques during one rotation of the body is small in comparison which its kinetic energy; the vehicle performs an almost free Euler motion whose parameters vary slowly in time. The motion of unoriented satellites is usually rotational.

In studying rotational motion it is often convenient to use the methods of perturbation theory, regarding the Euler motion as unperturbed. By averaging one can obtain the evolution equations describing the variation of the angular momentum vector and other parameters of the Euler motion. In this connection the simplest case is that of dynamic symmetry.

By similar methods it has been possible, in particular, to carry out fairly detailed investigations of the influence of gravitational, aerodynamic, and magnetic torques and certain other factors on satellite motion.

Further investigations require a more detailed determination of the influence of dissipative factors and a more detailed consideration of the general case (not assuming dynamic symmetry).

A space vehicle may oscillate about some unperturbed position due to perturbations. Such motions are called librational. An example is the moon, which maintains an almost unchanged position (turning one face permanently to Earth), performing only very small oscillations—librations. This natural stabilization is caused by the small deviation of the ellipsoid of inertia of the moon from a sphere, and the appearance of restoring torques due to the gravitational field of Earth when deviations from the position of equilibrium occur.

The same effect can be used to stabilize artificial satellites. The requisite satellite attitude stabilization in space can be achieved by providing the satellite with a special body—a stabilizer, to ensure gravitational stability of the satellite-stabilizer system. In order to prevent the satellite and to ensure oscillations sufficiently small in amplitude, the perturbations forming upon separation from the carrier must be suppressed and damping introduced in the satellite-stabilizer system. Such damping can be achieved, for instance, by introducing relative mobility of satellite and stabilizer and dissipative coupling in relative displacements.

For a correct interpretation of the readings of scientific instruments mounted on unoriented research satellites, it is often necessary to know the orientation of the satellite at the time of measurement. The required information is obtainable from the readings of both the research instruments and special orientation sensors. However, in many cases this information is incomplete, and direct determination of the atitude of the satellite is not sufficiently accurate.

In this case considerable progress can be achieved by processing the available information statistically, taking the known regularities of the motion into account. The equations of motion can take into account the effect of both known and incompletely known factors, where the uncertainty in the latter depends on unknown parameters to be determined along with the parameters of motion. The resulting problem of determining the motion and calculating the orientation of the satellite at any time is analogous to that of processing trajectory measurements, and is an integral part of the space research.

Apart from its direct utilitarian value, the problem of determining the motion about the center of mass from measurements is extremely important for studying the regularities of the motion, ascertaining dynamic effects, determining perturbing torques and their influence, and constructing a sound dynamic scheme of the space vehicle.

An important field in the study of space motion, though unfortunately very little studied, is that studying the influence of fluid contents.

One interesting problem concerning fluid-filled cavities is that of the decay and evolution of rotational motions due to dissipation caused by a viscous fluid. If the fluid has a free surface the motion is very complicated. The fairly well-developed problem of small oscillations of a heavy fluid in a vessel under small perturbations of the free plane surface becomes meaningless in conditions of weightlessness. Effects such as surface tension become important.

All these interesting problems, on the border line between hydrodynamics and rigid body dynamics, are still open.

This section in no way exhausts the diversity of mechanical problems arising in the motion of space vehicles about the center of mass; our modest purpose was a brief outline of some of them.

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EXPLANATORY LIST OF ABBREVIATED NAMES OF USSR INSTITUTIONS APPEARING IN THIS TEXT

Abbreviation	Full name (transliterated)	Translation
GGU	Gor'kovskii Gosudarstvennyi Universitet im. N.I. Lobachevskogo	N.I. Lobachevski State University in Gorki
ITA	Institut Teoreticheskoi Astronomii (Akademii Nauk SSSR)	Institute of Theoretical Astronomy (of the Academy of Sciences of the USSR)
KAI	Kazanskii Aviatsionnyi Institut	Kazan Aviation Institute
LGU	Leningradskii Gosudarstvennyi Universitet	Leningrad State University
VZPI	Vsesoyuznyi Zaochnyi Politekhnicheskii Institut	All-Union Correspondence Polytechnic Institute

Table of Contents

Preface v

ANALYTICAL MECHANICS. STABILITY OF MOTION

I.S. Arzhanykh. Development of the theory of canonical equations of analytical mechanics 1

V.I. Arnol'd. Stability and instability of dynamic systems with many degrees of freedom . . . . 11.

G.V. Aronovich and N.A. Kartvelishvili. Application of stability theory to static and dynamic stability problems of power systems 15

F.R. Gantmakher and V.A. Yakubovich. Absolute stability of nonlinear control systems 27

N.A. Kil'chevskii. Continuum problems of analytical mechanics 67

N.N. Krasovskii. Problems of controllability, observability and stabilizability of dynamic systems 82

A.M. Letov. Optimal control and stability 101

V.M. Matrosov. Development of the method of Lyapunov functions in stability theory 124

G.N. Mikishev and B.I. Rabinovich. Some problems in dynamic stability analysis of mechanical systems with deformable elements 141

Yu.I. Neimark and N.A. Fufaev. Equations of motion and the role of small parameters in the dynamics of nonholonomic systems 146

V.S. Novoselov. Analytical dynamics and optimum transfer 157

V.V. Rumyantsev. Investigation of the stability of motion of solid bodies with liquid-filled cavities 170

S.N. Shimanov. Stability of systems with delay 189

CELESTIAL BALLISTICS

G.L. Grodzovskii, Yu.N. Ivanov, and V.V. Tokarev. Optimization problems in the mechanics of low-thrust space flight 201

V.A. Sarychev. Gravity attitude stabilization of satellites 222

G.N. Duboshin and D.E. Okhotsimskii. Some problems of astrodynamics. 233

List of Abbreviations 239